

A Cotangent Bundle Slice Theorem

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Abstract

This article concerns cotangent-lifted Lie group actions; our goal is to find local and “semi-global” normal forms for these and associated structures. Our main result is a constructive cotangent bundle slice theorem that extends the Hamiltonian slice theorem of Marle [Mar85] and Guillemin and Sternberg [GS84]. The result applies to all proper cotangent-lifted actions, around points with fully-isotropic momentum values.

We also present a “tangent-level” commuting reduction result and use it to characterise the symplectic normal space of any cotangent-lifted action. In two special cases, we arrive at splittings of the symplectic normal space. One of these cases is when the configuration isotropy group is contained in the momentum isotropy group; in this case, our splitting generalises that given for free actions by Montgomery et al. [MMR84]. The other case includes all relative equilibria of simple mechanical systems. In both of these special cases, the new splitting leads to a refinement of the so-called *reconstruction equations* or *bundle equations* [Ort98, OR02b, RWL02]. We also note cotangent-bundle-specific local normal forms for symplectic reduced spaces.

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1 Introduction

This article concerns cotangent-lifted actions of a Lie group G on a cotangent bundle T^*Q . We are motivated in part by the role of such actions as groups of symmetries of Hamiltonian systems with cotangent bundle phase spaces. Nonetheless, this article is primarily geometric, the exception being the discussion of the reconstruction equations (bundle equations) at the end of Section 4.

When G acts freely and properly, it is well known that one can *reduce* T^*Q by the G action to give a lower-dimensional symplectic manifold (see Theorem 2.2). The reduced manifold inherits some cotangent-bundle structure [AM78, Mars92, MP00], and it sometimes actually *is* a cotangent bundle (see for example Theorem 2.3). In Hamiltonian systems, the solutions of the system on the original space project to solutions of a new Hamiltonian system on the reduced phase space.

One would like to generalise this picture to arbitrary proper group actions, not necessarily free. This problem of *singular reduction* has been addressed with success in the symplectic category [SL91, BL97] and more recently for the special case of cotangent bundles [CS01, PRS03]. But the symplectic reduced spaces are in general not smooth, and our understanding of the inherited cotangent bundle structure is far from complete.

A different but related approach is to ask: to what degree can we factor out the symmetry while not losing smoothness? Slice theorems are an answer to this question. For any *free* proper action of G on a manifold M , the slice theorem of Palais (Theorem 2.4) says that every point $z \in M$ has a neighbourhood G -equivariantly isomorphic to a space $G \times S$, where S is some submanifold of M transverse to the G orbit and the G action on $G \times S$ is $g' \cdot (g, s) = (g'g, s)$. This local model of the action of G on M is actually “semi-global” in the sense that it is global in the G direction but local in the transverse direction. For general proper actions, the model space is not $G \times S$ but a *twisted* space $G \times_{G_z} S$, where G_z is the isotropy group of the point z . For symplectic actions, the Hamiltonian slice theorem of Marle [Mar85] and Guillemin and Sternberg [GS84] (Theorem 2.6) gives a model space of this kind and a G -equivariant *symplectic* diffeomorphism. This theorem is a fundamental tool in the study of Hamiltonian systems with symmetry: it has found applications to singular reduction [SL91, BL97, Ort98], and to many dynamical questions involving stability, bifurcation and persistence in the neighbourhood of relative equilibria and relative periodic orbits [Mo97, Mo97a, RdSD97, LS98, OR99, RWL02, Ort03, OR02a, WR02].

The main aim of the present article is to extend the Hamiltonian slice theorem in the context of cotangent bundles. We succeed in doing so around points with fully isotropic momentum values (Theorem 5.6). Our new result extends that of Marle, Guillemin and Sternberg in three ways. First, it involves a new cotangent-bundle-specific splitting of the symplectic normal space. Second, it is constructive, up to a Riemannian exponential map. In particular, we do not use the Constant Rank Embedding Theorem or Darboux’s Theorem. Third, our construction has a unique property (see Lemma 5.3).

We begin with a summary of the relevant background material, including some reduction results and slice theorems. In Section 3 we summarise regular and singular commuting symplectic reduction, and introduce a new “tangent level” commuting reduction result that

works at the level of symplectic normal spaces (Theorem 3.5).

In Section 4 we analyse the symplectic normal space at $z \in T_q^*Q$ for a cotangent-lifted action of G on T^*Q . We first apply Palais' slice theorem in the configuration space and then cotangent-lift the resulting diffeomorphism to give a local symplectic diffeomorphism from T^*Q to $T^*(G \times_{G_q} A)$, where A is a G_z -invariant complement to the tangent to the orbit $G \cdot q$. We then “unroll” this space by considering the untwisted product $T^*(G \times A)$. We note that there are two obvious commuting actions on this space, namely cotangent lifts of left multiplication by G and twist by G_q . Applying tangent-level commuting reduction leads to our characterisation of the symplectic normal space N_s of the original G action on T^*Q , Theorem 4.9. Corollaries 4.10 and 4.13 give splittings of N_s in two special cases: $G_z \subset G_\mu$, where μ is the momentum value of z ; and z “purely in the group direction”, meaning $z|_A = 0$. We note consequences of these results for singular reduction and for the reconstruction equations (bundle equations).

In Section 5, we prove the cotangent bundle slice theorem. We begin with the observation that, when $G_\mu = G$, our new splitting $N_s \cong T^*B$ (for a certain subspace B of A) implies that the model space in the Hamiltonian slice theorem is $G \times_{G_z} ((\mathfrak{g}/\mathfrak{g}_z)^* \oplus T^*B)$, which is a G_z -reduced space of $T^*(G \times B)$. The problem of proving a constructive Hamiltonian slice theorem thus reduced in this case to that of finding a certain symplectic local diffeomorphism from $T^*(G \times_{G_z} B)$ to $T^*(G \times_{G_q} A)$. We proceed to find a suitable symplectomorphism, using two alternative methods. The first method is more “brute-force” and gives an explicit formula in coordinates; the second method is to re-arrange the problem so that a cotangent-lift can be used. We end with a simple example, $SO(3)$ acting on $T^*\mathbf{R}^3$.

Most of the results in this article first appeared in the author's PhD thesis [Sch02], where more detailed proofs of some results appear.

Some assumptions and notation: We consider only proper actions on finite-dimensional manifolds. All cotangent bundles are given the standard cotangent bundle symplectic form; in particular, no magnetic terms appear. All group actions are left actions. The Lie algebra of a Lie group will always be denoted by the corresponding fraktur letter.

2 Preliminaries

We summarise relevant basic facts on Lie group symmetries symplectic reduction and slice theorems. This material is well-known; good general references are [AM78, DK99, CB97, OR04].

Lie Group Actions Let G be a Lie group, with Lie algebra \mathfrak{g} , and consider a smooth left action Φ of G on a manifold M ; we write $g \cdot z = \Phi_g(z) = \Phi(g, z)$. For every $\xi \in \mathfrak{g}$, the *infinitesimal generator* of ξ is the vector field ξ_M defined by $\xi_M(z) = \frac{d}{dt} \exp(t\xi) \cdot z|_{t=0}$. We will also write $\xi_M(z)$ as $\xi \cdot z$, and refer to the map $(\xi, z) \mapsto \xi \cdot z$ as the *infinitesimal action* of \mathfrak{g} on M .

The action Φ is *proper* if the map $(g, z) \mapsto (z, g \cdot z)$ is proper (i.e. the preimage of every compact set is compact). It is easily shown that all proper actions on manifolds have

the following property (often used as a definition of properness): given any convergent sequences $\{z_i\}$ and $\{g_i \cdot z_i\}$, the sequence $\{g_i\}$ has a convergent subsequence.

If G acts properly and freely on M , then M/G has a unique smooth structure such that $\pi_G : M \rightarrow M/G$ is a submersion (in fact, π_G is a principal bundle). One useful consequence is that for every $z \in M$, we have $\ker T_z \pi_G = T_z (G \cdot z) = \mathfrak{g} \cdot z$.

The *isotropy subgroup* of a point $z \in M$ is $G_z := \{g \in G \mid g \cdot z = z\}$. The isotropy subgroups are always Lie subgroups, since they are clearly closed. An action is *free* if all of the isotropy subgroups G_z are trivial.

A key elementary property of proper actions is that all isotropy subgroups are compact. This property is often used to produce H -invariant structures by averaging over some isotropy group H .

Momentum Maps Now suppose G acts symplectically on a symplectic manifold (M, ω) . Recall that any function $F : M \rightarrow \mathbf{R}$ defines a Hamiltonian vector field X_F by $i_{X_F} \omega = dF$, in other words $\omega(X_F(z), v) = dF(v)$ for every $v \in T_z^* M$. A *momentum map* is a function $J : M \rightarrow \mathfrak{g}^*$ satisfying $X_{J_\xi} = \xi_M$ for every $\xi \in \mathfrak{g}$, where $J_\xi : M \rightarrow \mathbf{R}$ is defined by $J_\xi(z) = \langle J(z), \xi \rangle$. The Ad^* action of G on \mathfrak{g}^* is given by $g \cdot \nu = \text{Ad}_{g^{-1}}^* \nu := (\text{Ad}_{g^{-1}})^* \nu$, where Ad_g is the adjoint operator. If the G action has an Ad^* -equivariant momentum map J , then it is called *globally Hamiltonian*. Note that if $J(z) = \mu$ and G_μ is the isotropy group of μ with respect to the coadjoint action and J is Ad^* -equivariant then $G_z \subset G_\mu$.

We note here an important momentum map, that of the coadjoint action of G on any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$. The Kostant-Kirillov-Souriau (KKS) symplectic forms on the coadjoint orbit are

$$\omega_{\mathcal{O}}^\pm(\nu)(\xi \cdot \nu, \eta \cdot \nu) = \pm \langle \nu, [\xi, \eta] \rangle, \quad (1)$$

where $\xi \cdot \nu = -\text{ad}_\xi^* \nu$, the infinitesimal generator of the coadjoint action of G on \mathfrak{g}^* . The momentum map of the coadjoint action of G on \mathcal{O} with respect to $\omega_{\mathcal{O}}^\pm$ is $J_{\mathcal{O}}(\nu) = \pm \nu$.

Lifted Actions on (Co-)tangent Bundles. Every cotangent bundle T^*Q has a canonical symplectic form, given in given local coordinates by $\omega = dq^i \wedge dp_i$. The space Q is called the *configuration space* or *base space*. The *tangent lift* of any action $\Phi : G \times Q \rightarrow Q$ is the action of G on TQ given by $g \cdot v = T\Phi_g(v)$. The *cotangent lift* is the action of G on T^*Q given by $g \cdot \alpha = (T\Phi_{g^{-1}})^* \alpha$. Here are some key elementary facts about lifted actions:

1. The tangent or cotangent lift of a proper (resp. free) action is proper (resp. free).
2. Every cotangent-lifted action on T^*Q is symplectic with respect to the canonical symplectic form on T^*Q .
3. Every cotangent-lifted action has an Ad^* -equivariant momentum map given by

$$\langle J(\alpha_q), \xi \rangle = \langle \alpha_q, \xi \cdot q \rangle.$$

(When we refer to “the” momentum map for such an action, this is the one we mean.)

We now consider the special case where Q is a vector space. Any G action on a vector space V induces an *inverse dual* (or *contragredient*) action of G on V^* . Specifically, if $\Psi : G \times V \rightarrow V$ is the G action on V , then the G action on V^* is $g \cdot \alpha = (\Psi_{g^{-1}})^* \cdot \alpha$. Identifying T^*V with $V \oplus V^*$, the cotangent- lifted action of V on T^*V is $g \cdot (a, \alpha) = (g \cdot a, g \cdot \alpha)$, where the action on the second component is the inverse dual action. The infinitesimal action of G on V^* is $\langle \eta \cdot \alpha, a \rangle = \langle \alpha, -\eta \cdot a \rangle$.

Note that if we identify V^{**} with V then the inverse dual of the inverse dual of an action is the original action.

We introduce the diamond notation of Holm et al. [HMR98], adding an optional subscript to specify the Lie algebra of the symmetry group or some linear subspace of it. For every $a \in V$, $\alpha \in V^*$, and any subspace \mathfrak{l} of \mathfrak{g} , we define $a \diamond \alpha \in \mathfrak{g}^*$ and $a \diamond_{\mathfrak{l}} \alpha \in \mathfrak{l}^*$ by

$$\begin{aligned} \langle a \diamond \alpha, \xi \rangle &= \langle \alpha, \xi \cdot a \rangle \text{ for all } \xi \in \mathfrak{g}, \text{ and} \\ a \diamond_{\mathfrak{l}} \alpha &= a \diamond \alpha|_{\mathfrak{l}} \end{aligned} \tag{2}$$

We will have occasion to use the isomorphism $T^*V \cong V \oplus V^* \cong V^{**} \oplus V^* \cong T^*V^*$, so we point out that $\alpha \diamond_{\mathfrak{g}} a = -a \diamond_{\mathfrak{g}} \alpha$ for any $a \in V$ and $\alpha \in V^*$.

The momentum map for the cotangent-lifted G action on $T^*V \cong V \oplus V^*$ is the map $J_V : V \rightarrow \mathfrak{g}^*$ given by $J_V(a, \alpha) = a \diamond_{\mathfrak{g}} \alpha$.

Symplectic Reduction We now present three symplectic reduction theorems. The simplest one is linear:

Theorem 2.1 *Let ω be a skew-symmetric bilinear form on a vector space V . Then V/V^ω has a symplectic form given by $\omega_{red}(u + V^\omega, v + V^\omega) = \omega(u, v)$.*

For symplectic manifolds, one of the simplest forms of reduction is the following, which is a slight simplification of a version due to Marsden and Weinstein [MW74].

Theorem 2.2 (Regular “Point” Symplectic Reduction) *Let G act freely, properly and symplectically on a symplectic manifold (M, ω) , with Ad^* -equivariant momentum map $J : M \rightarrow \mathfrak{g}^*$, and let $\mu \in \mathfrak{g}^*$. Then the reduced space $J^{-1}(\mu)/G_\mu$ has a symplectic form ω_μ uniquely defined by $\pi_\mu^* \omega_\mu = i_\mu^* \omega$, where $\pi_\mu : J^{-1}(\mu) \rightarrow J^{-1}(\mu)/G_\mu$ and $i_\mu : J^{-1}(\mu) \rightarrow M$ is inclusion.*

If the symplectic manifold is a cotangent bundle, then the reduced space takes a special form. The following version of cotangent bundle reduction is a special case of results by Satzer and Marsden (see [AM78]).

Theorem 2.3 (Regular “Point” Cotangent Bundle Reduction at Zero) *Let G act freely and properly by cotangent lifts on T^*Q , and let J be the momentum map of the G action (with respect to the canonical symplectic form on T^*Q). Then $J^{-1}(0)$ is a smooth submanifold of Q . Let $\pi_G : Q \rightarrow Q/G$ be projection. Define the map $\varphi : J^{-1}(0) \rightarrow T^*(Q/G)$ by, for every $q \in Q$ and every $p \in T_q^*Q$ and $v \in T_q Q$,*

$$\langle \varphi(p), T\pi_G(v) \rangle = \langle p, v \rangle.$$

Then φ is a G -invariant surjective submersion and drops to a symplectomorphism (i.e. symplectic diffeomorphism)

$$\bar{\varphi} : J^{-1}(0)/G \rightarrow T^*(Q/G),$$

where the left-hand side has the reduced symplectic form corresponding to the canonical symplectic form on T^*Q , and $T^*(Q/G)$ has the standard symplectic form.

The map φ is a sort of push-forward, though π_G is not injective. Note that φ is “injective mod G ”, meaning that $\varphi(z_1) = \varphi(z_2)$ if and only if $z_1 = g \cdot z_2$ for some $g \in G$.

The Symplectic Normal Space and the Witt-Artin Decomposition Let G be a Lie group acting symplectically and properly on (M, ω) , and let $z \in M$. The Witt-Artin decomposition is a splitting

$$T_z M = T_1 \oplus (T_0 \oplus N_0) \oplus N_1 \quad (3)$$

such that $T_1 \oplus T_0 = \mathfrak{g} \cdot z$ (so “T” is for “tangent” and “N” is for “normal”), and $N_1 \oplus T_0 = (\mathfrak{g} \cdot z)^\omega$ (the symplectic complement to $\mathfrak{g} \cdot z$) and each of the three spaces $T_1, (T_0 \oplus N_0)$ and N_1 is a symplectic subspace of $(T_z M, \omega(z))$. The decomposition can be chosen to be G_z -invariant, where G_z is the isotropy group of z .

We now define the decomposition, which is not unique, due to a choice of complements. Though it can be defined more generally, we will assume the existence of an Ad^* -equivariant momentum map J . Let $\mu = J(z)$ and let G_μ be the isotropy group of μ under the coadjoint action. Let \mathfrak{g} and \mathfrak{g}_μ be the Lie algebras of G and G_μ . The well-known “Reduction Lemma” states that $(\mathfrak{g} \cdot z)^\omega = \ker dJ(z)$ and $\mathfrak{g} \cdot z \cap (\mathfrak{g} \cdot z)^\omega = \mathfrak{g}_\mu \cdot z$. We define $T_0 = \mathfrak{g}_\mu \cdot z$. Let T_1 be a G_z -invariant complement to T_0 , which always exists since G_z is compact. Similarly, let N_1 be a G_z -invariant complement to $\mathfrak{g}_\mu \cdot z$ in $\ker dJ(z)$. Since the kernel of ω restricted to either $\mathfrak{g} \cdot z$ or $\ker dJ(z)$ is $\mathfrak{g}_\mu \cdot z$, the subspaces T_1 and N_1 are symplectic. It can be shown that T_0 is a Lagrangian subspace of $(T_1 \oplus N_1)^\omega$ and there exists a Lagrangian subspace N_0 of $(T_1 \oplus N_1)^\omega$ such that $(T_1 \oplus N_1)^\omega = T_0 \oplus N_0$ is a G_z -invariant splitting. Then it can be shown that the Witt-Artin decomposition (Equation 3) has the properties stated above, and that there is a G_z -invariant isomorphism of N_0 with $(\mathfrak{g}_\mu/\mathfrak{g}_z)^*$. See [OR04] for proofs and further discussion.

The space N_1 is often called the *symplectic normal space* to the group orbit through z , but we will reserve this term for the isomorphic space

$$N_s = \ker dJ(z)/\mathfrak{g}_\mu \cdot z,$$

which has a reduced symplectic bilinear form defined as in Theorem 2.1. Since $G_z \subset G_\mu$, it is easy to show that the tangent-lifted action of G_z on $T_z M$ leaves $\ker dJ(z)$ invariant and descends to a symplectic action on N_s given by

$$h \cdot (v + \mathfrak{g}_\mu \cdot z) = T\Phi_h(v) + \mathfrak{g}_\mu \cdot z, \quad (4)$$

and that N_s is G_z -equivariantly symplectomorphic to N_1 for any choice of N_1 as above. Note that in the case of free actions, N_s is symplectomorphic to the tangent space at $[z]_{G_\mu}$ to the reduced space $J^{-1}(\mu)/G_\mu$.

Slice Theorems While symplectic reduction can be generalised to singular momentum values, the resulting spaces are in general not smooth [SL91, BL97, OR04]. It thus makes sense to consider a related question: how far can we factor out a symmetry without losing smoothness? Slice theorems are local answers to this question (or “semilocal”, since the model spaces are “global in the group direction”). They say that such a space is locally isomorphic to a twisted product of the group and a “slice” transverse to the group orbit. In the case of a free action, there is no twisting; in general, the twisting is by the isotropy group of the point at which the local model is based.

We first define twisted products, slices and tubes and state a version of Palais’ slice theorem for the category of Lie group symmetries on manifolds. We then define the symplectic normal space and state the Hamiltonian Slice Theorem of Marle, Guillemin and Sternberg. Our new cotangent bundle slice theorem will be presented later as Theorem 5.6.

Let H be a Lie subgroup of a Lie group G , and N is a manifold on which H acts. Consider the following two left actions on $G \times N$:

$$\begin{aligned} K \text{ acts by } \textit{twisting} : k \cdot (g, n) &= (gk^{-1}, k \cdot n) \\ G \text{ acts by } \textit{left multiplication} : g' \cdot (g, n) &= (g'g, n). \end{aligned} \tag{5}$$

These actions are easily seen to be free and proper. The *twisted product* $G \times_H N$ is the quotient of $G \times N$ by the *twist action* $h \cdot (g, n) = (gh^{-1}, h \cdot n)$. It is a smooth manifold (since the twist action is free); in fact $G \times_H N \rightarrow G/H$ is the vector bundle associated to the H action on N . The *left multiplication action* of G on itself commutes with the twist action and drops to a smooth G action on $G \times_H N$, namely $g' \cdot [g, n]_H = [g'g, n]_H$.

Now consider a G action on a manifold M , and a point $z \in M$, and let $H = G_z$ be the isotropy subgroup of z . A *tube* for the G action at z is a G -equivariant diffeomorphism from some twisted product $G \times_H N$ to an open neighbourhood of z in M , that maps $[e, 0]_H$ to z . The space N may be embedded in $G \times_H N$ as $\{[e, n]_H : n \in N\}$; the image of the latter by the tube is called a *slice*. A *slice theorem* is a theorem that guarantees the existence of a tube under certain conditions. Palais [P61] was the first to prove a slice theorem for proper actions. Many smooth versions of his original theorem are in common use. A proof of the following version appeared in an appendix to an earlier version of the present article, but this appendix has now been moved to the author’s website due to space considerations and because the proof has recently appeared (with permission) in [OR04].

Theorem 2.4 (“Palais’ Slice Theorem”) *Let G be a Lie group acting properly and smoothly on a manifold M , and let $z \in M$. Let $H = G_z$ be the isotropy group of z , and let N be any H -invariant complement to $\mathfrak{g} \cdot z$. Choose a local H -invariant Riemannian metric around z (such a metric always exists), and let \exp_z be the corresponding Riemannian exponential based at z . Then there exists an H -invariant neighbourhood U of 0 in N such*

that the map

$$\begin{aligned}\tau : G \times_H U &\rightarrow M \\ [g, n]_H &\longmapsto g \cdot \exp_z n\end{aligned}$$

is a tube for the G action at z .

Because of this theorem, an H -invariant complement to $\mathfrak{g} \cdot z$ is sometimes called a *linear slice* to the G action at z .

In the case of a linear action, we can replace “ $\exp_z n$ ” with “ $z + n$ ” in the above statement. Indeed, it is easy to prove that a constant H -invariant Riemannian metric always exists; the corresponding exponential will be the map $n \mapsto z + n$. Alternatively, a special version of the slice theorem for linear actions can be proven directly; in doing so, one notes that the map τ will be a tube whenever the neighbourhood U is small enough that τ is injective. Thus we have the following result.

Theorem 2.5 (Slice theorem for linear actions) *Let G be a Lie group acting properly, smoothly and linearly on \mathbf{R}^n and let $z \in \mathbf{R}^n$. Let $H = G_z$ be the isotropy group of z , and let N be any H -invariant complement to $\mathfrak{g} \cdot z$. Then there exists an H -invariant neighbourhood U of 0 in N such that the map*

$$\begin{aligned}\tau : G \times_H U &\rightarrow \mathbf{R}^n \\ [g, n]_H &\longmapsto g \cdot (z + n)\end{aligned}$$

is injective. Given any such U , the map τ is a tube for the G action at z .

The Hamiltonian Slice Theorem Now suppose that G acts *symplectically* on a symplectic manifold (M, ω) . We would like tube τ to be symplectic. One could obviously pull back the symplectic form on M by the diffeomorphism given by Palais’ slice theorem, but the resulting symplectic structure need not be simple or “natural”. The Hamiltonian slice theorem defines a “natural” symplectic form on a space $G \times_H N$ and then shows that this space is G -equivariantly locally symplectomorphic with M .

The Hamiltonian slice theorem was first proven by Marle [Mar85] and Guillemin and Sternberg [GS84], for compact groups G and extended to proper actions of arbitrary groups by Bates and Lerman [BL97]. The Hamiltonian slice theorem is called “Hamiltonian” because it assumes that the G action is globally Hamiltonian, meaning that it has a globally defined Ad^* -equivariant momentum map J . We note that this assumption has been removed by Ortega and Ratiu [OR02b] and Scheerer and Wulff [SW01]. The Hamiltonian slice theorem is sufficiently general for the present article, since all cotangent-lifted actions have an Ad^* -equivariant momentum map. We now present the Hamiltonian slice theorem, following closely the presentations in [Ort98] and [SL91], to which we refer the reader for details and proofs.

We are assuming that G acts symplectically and properly on (M, ω) , with Ad^* -equivariant momentum map J . Let $z \in M$ and let $H = G_z$ be the isotropy group of z . Let $\mu = J(z)$

and let G_μ be the isotropy group of μ under the coadjoint action. Note that $H \subset G_\mu$, by the Ad^* -equivariance of J . Let \mathfrak{h} , \mathfrak{g} and \mathfrak{g}_μ be the Lie groups of H , G and G_μ respectively. Let \mathfrak{m} be an H -invariant complement to \mathfrak{h} in \mathfrak{g}_μ . Recall from above that the *symplectic normal space* at z is $N_s = \ker T_z J / \mathfrak{g}_\mu \cdot z$ and that there is a reduced symplectic form and a natural H action on N_s . This H action and the coadjoint action of H on \mathfrak{m}^* define an H action on $\mathfrak{m}^* \oplus N_s$, allowing us to define the twisted product

$$G \times_H (\mathfrak{m}^* \oplus N_s).$$

This will be the model space of the Hamiltonian slice theorem. Recall from above that there is an isomorphism $\mathfrak{m}^* \cong N_0$ and a symplectomorphism $N_s \cong N_1$, both H -equivariant, where N_0 and N_1 are components in the Witt decomposition. The sum $N_0 \oplus N_1$ is linear slice at z . Thus the model space $G \times_H (\mathfrak{m}^* \oplus N_s)$ can be considered to be a special case of the model space $G \times_H N$ in Theorem 2.4.

We now define the symplectic form on the space $G \times_H (\mathfrak{m}^* \oplus N_s)$, beginning with a presymplectic form (i.e., a closed two-form) on $G \times \mathfrak{g}_\mu^* \oplus N_s$. First, let Ω_c be the pull-back to $G \times \mathfrak{g}_\mu^*$ of the canonical form on T^*G by the map $G \times \mathfrak{g}_\mu^* \rightarrow T^*G$, $(g, \nu) \mapsto TL_{g^{-1}}^* \nu$. Second, let Ω_μ be the pull-back by the map $G \times \mathfrak{g}_\mu^* \rightarrow \mathcal{O}_\mu$, $(g, \nu) \mapsto \text{Ad}_{g^{-1}}^* \mu$, of the KKS symplectic form $\omega_{\mathcal{O}_\mu}^+$ (defined in Eq. 1). Third, let ω_{N_s} be the reduced symplectic bilinear form on N_s . The sum $\Omega_Z = \Omega_c + \Omega_\mu + \Omega_{N_s}$ is a presymplectic form on $Z = G \times (\mathfrak{g}_\mu^* \oplus N_s)$.

Consider the twist action of H on Z corresponding to the coadjoint action of H on \mathfrak{g}_μ^* and the H action on N_s inherited from the lifted action of G on $T_z M$. The H -action on N_s has an H -equivariant momentum map J_{N_s} (as does any linear symplectic action). One can check that the twist action of H on Z is globally Hamiltonian with respect to Ω_Z , with momentum map $J_H : (g, \sigma, v) \mapsto J_{N_s}(v) - \sigma|_{\mathfrak{h}}$. If we identify \mathfrak{m}^* with $\mathfrak{k}^\circ \subset \mathfrak{g}_\mu^*$, then the following map is well-defined,

$$\begin{aligned} l : G \times \mathfrak{m}^* \oplus N_s &\longrightarrow J_H^{-1}(0) \subset Z \\ (g, \sigma, v) &\longmapsto (g, \sigma + J_{N_s}(v), v) \end{aligned} \quad (6)$$

It is clearly an H -equivariant diffeomorphism (with respect to twist action of H on $G \times \mathfrak{m}^* \oplus N_s$ defined earlier). This map descends to a diffeomorphism L defined by the following commutative diagram,

$$\begin{array}{ccc} G \times (\mathfrak{m}^* \oplus N_s) & \xrightarrow{l} & J_H^{-1}(0) \subset Z = G \times (\mathfrak{g}_\mu^* \oplus N_s) \\ \downarrow \pi_H & & \downarrow \pi_{Z,H} \\ Y = G \times_H (\mathfrak{m}^* \oplus N_s) & \xrightarrow[\cong]{L} & J_H^{-1}(0) / H, \end{array} \quad (7)$$

where π_H and $\pi_{Z,H}$ are the obvious projections.

We define the presymplectic form ω_Y on $G \times_H (\mathfrak{m}^* \oplus N_s)$ as the pull-back by L of the reduced presymplectic form on $J_H^{-1}(0) / H$ corresponding to Ω_Z . It can be shown that there exists a G -invariant neighbourhood of $[e, 0, 0]_H$ in Y in which ω_Y is symplectic.

Finally, note that there is left G -action on Y given by $g' \cdot [g, \sigma, v]_H = [g'g, \sigma, v]_H$. It is easy to check that this is symplectic with respect to ω_Y . We can now state the Hamiltonian Slice Theorem, also known as the Marle-Guillemin-Sternberg Normal Form.

Theorem 2.6 (Hamiltonian Slice Theorem) *In the above context, there exists a symplectic tube from $Y = G \times_H (\mathfrak{m}^* \oplus N_s)$ to M that maps $[e, 0, 0]_H$ to z .*

It can be shown that the momentum map of the G action on Y is

$$J_Y([g, \sigma, v]_H) = \text{Ad}_{g^{-1}}^*(\mu + \sigma + J_{N_s}(v))$$

3 Commuting reduction

In this section we consider a manifold with two commuting symplectic actions. We first review regular and singular commuting reduction and then introduce a new “tangent-level version” of commuting reduction, which we will use in the next section in our analysis of the symplectic normal space of a cotangent-lifted action.

We have already seen an example of commuting symplectic actions in the presentation of the Hamiltonian slice theorem: the G and H actions on the manifold $G \times (\mathfrak{g}_\mu^* \oplus N_s)$ (see Equation 7). In this context, commuting reduction leads to a singular local normal form for a symplectic reduced space, Theorem 3.3. A second example of commuting symplectic actions, key to the rest of this article, will appear in the next section: a bundle $T^*(G \times A)$ with the cotangent lifts of the left multiplication action of G and the twist action of a subgroup K of G . Commuting reduction in this context leads to a cotangent-bundle specific local normal form for a symplectic reduced space, Theorem 4.8. “Tangent-level reduction” in this context will be used to characterise the symplectic normal space of a cotangent-lifted action: see Theorem 4.9 and following results.

Let G and K be Lie groups acting symplectically and properly on a symplectic manifold M , with equivariant momentum maps J_G and J_K respectively, and suppose that the actions commute. Let $\mu \in \mathfrak{g}^*$ and $\nu \in \mathfrak{k}^*$. The idea of commuting reduction is to first reduce by the K action (say) and then reduce the K -reduced space by the induced G action; and then switch the order, reducing first by G and then by K . Under very general conditions, the two doubly-reduced spaces are isomorphic. We first state the “regular version” of commuting reduction, due to Marsden and Weinstein [MW74]; the key assumption here is that all of the group actions are free.

Theorem 3.1 (Regular Commuting Reduction) *In the above context, suppose that G and K act freely and J_K is G -invariant and J_G is K -invariant. Then G induces a symplectic action on $M_\nu := J_K^{-1}(\nu)/K_\nu$ with equivariant momentum map $J_{\bar{G}}$ determined by $J_{\bar{G}} \circ \pi_{K_\nu} = J_G$ (where $\pi_{K_\nu} : M \rightarrow M/K_\nu$ is projection, and both sides of the equation are restricted to $J_K^{-1}(\nu)$). If the reduced G action is free, then the reduced space for this action at μ is symplectomorphic to the reduction of M at (μ, ν) by the product action of $G \times K$.*

Note that applying this theorem a second time, with the roles of G and K reversed, shows that the reduced space at ν for the action of K on $J_G^{-1}(\mu)/G_\mu$ is symplectomorphic to the reduced space at μ for the action of G on M_ν .

Sjamaar and Lerman [SL91], working with reduction at zero of compact group actions, showed that a similar result holds even if the actions are not free. In this case, the reduced

spaces need not be smooth manifolds, but are Poisson varieties. In the general case, for proper actions and arbitrary momentum values, we need to add the hypotheses that G_μ and K_ν are compact and that \mathcal{O}_μ and \mathcal{O}_ν are locally closed, the latter for reasons discussed in [MMOPR].

Theorem 3.2 (Singular Commuting Reduction) *In the above context, suppose that J_K is G -invariant, J_G is K -invariant, G_μ and K_ν are compact and the coadjoint orbits \mathcal{O}_μ and \mathcal{O}_ν are locally closed. Then G induces a Poisson action on $M_\nu = J_K^{-1}(\nu)/K_\nu$, with equivariant momentum map $J_{\bar{G}}$ determined by $J_{\bar{G}} \circ \pi_{K_\nu} = J_G$. The reduced space for the action of G on M_ν at μ is Poisson diffeomorphic to the reduction of M at (μ, ν) by the product action of $G \times K$.*

It follows that the reduced space at ν for the action of K on $J_G^{-1}(\mu)/G_\mu$ is Poisson diffeomorphic to the reduced space at μ for the action of G on M_ν .

The Hamiltonian Slice Theorem (Theorem 2.6), together with singular commuting reduction, applied to the G and H actions on $G \times (\mathfrak{g}_\mu^* \oplus N_s)$ (see Equation 7), can be used to deduce the following local normal form for a symplectic reduced space (when G_μ is compact). The result was first published by Sjamaar and Lerman [SL91] for $\mu = 0$; the general case is due to Bates and Lerman [BL97]. The proof given in [BL97] does not use a commuting reduction theorem and does not require G_μ compact.

Theorem 3.3 *Let G act properly on the symplectic manifold (M, ω) with equivariant momentum map J . Let $z \in M$ and $H = G_z$ and $\mu = J(z)$, and let N_s be the symplectic normal space to $\mathfrak{g} \cdot z$. Assume that the coadjoint orbit \mathcal{O}_μ is locally closed. Then there is a local Poisson diffeomorphism between the reduced space $J^{-1}(\mu)/G_\mu$ and the reduced space at 0 for the H action on N_s .*

In the case of cotangent-lifted actions, our analysis of the symplectic normal space, in the next section, together with the above theorem comprise a cotangent-bundle-specific local normal form for symplectic reduced spaces, as we note later in Remark 4.15.

We now introduce another approach to singular commuting reduction, assuming that the original actions are free but not assuming that the quotient action on the once-reduced space is free. Recall that, in the case of a free action, the symplectic normal space “is” the tangent space to the reduced space. This observation suggests studying symplectic normal spaces in place of the possibly singular doubly-reduced spaces.

Since symplectic normal spaces are quotients, the following lemma and notation will be useful; the lemma is easily checked.

Lemma 3.4 *Let ω_A and ω_B be bilinear forms on vector spaces A and B , respectively. Suppose $f : A \rightarrow B$ satisfies $f^*\omega_B = \omega_A$. Then the quotient map $\bar{f} : A/\ker(\omega_A) \rightarrow B/\ker(\omega_B)$ is well-defined and injective. If f is surjective, then \bar{f} is bijective. If ω_A and ω_B are presymplectic (i.e. skew-symmetric) then \bar{f} is symplectic. Also, if $\bar{g} : B/\ker(\omega_B) \rightarrow C/\ker(\omega_C)$ is defined similarly then $\overline{f \circ g} = \bar{f} \circ \bar{g}$.*

Theorem 3.5 (“Tangent-level” commuting reduction) *Let G and K be free, symplectic, commuting actions on a symplectic manifold M , with momentum maps J_G and J_K respectively. Then the product action of $G \times K$ has momentum map given by $J_{G \times K}(x) = (J_G(x), J_K(x))$. Let $x \in M$ and $(\mu, \nu) = J_{G \times K}(x)$. The symplectic normal space at x for the product action is*

$$N_s(x) = \ker T_x J_{G \times K} / (\mathfrak{g}_\mu \cdot x + \mathfrak{k}_\nu \cdot x).$$

Suppose further that G acts properly and that J_G is Ad^ -equivariant and that $J_G^{-1}(\mu)$ is K -invariant. Then the quotient action of K on $J_G^{-1}(\mu)/G_\mu$ is symplectic with respect to the reduced symplectic form, and its momentum map $J_{\bar{K}}$ satisfies $J_{\bar{K}} \circ \pi_{G_\mu} = J_K|_{J_G^{-1}(\mu)}$ (where $\pi_{G_\mu} : J_G^{-1}(\mu) \rightarrow J_G^{-1}(\mu)/G_\mu$ is projection). The map $(g, k) \mapsto k$ is a Lie group isomorphism from $(G \times K)_x$ to $K_{[x]_{G_\mu}}$ (where $[x]_{G_\mu} = G_\mu \cdot x$). We identify these two groups and call them H . Let $N_s([x]_{G_\mu})$ be the symplectic normal space at $[x]_{G_\mu}$ for the K action on $J_G^{-1}(\mu)/G_\mu$. Let H act on each symplectic normal space, as in Equation 4. Then the following is an H -equivariant vector space symplectomorphism,*

$$\begin{aligned} \overline{T_x \pi_{G_\mu}} : N_s(x) &\rightarrow N_s([x]_{G_\mu}) \\ v + (\mathfrak{g}_\mu \cdot x + \mathfrak{k}_\nu \cdot x) &\mapsto T\pi_{G_\mu}(v) + (\mathfrak{k}_\nu \cdot [x]_{G_\mu}). \end{aligned}$$

Proof. It is easily verified that the product action has the given momentum map. Since $(G \times K)_{(\mu, \nu)} = G_\mu \times K_\nu$ and the actions commute, we have $(\mathfrak{g} \oplus \mathfrak{k})_{(\mu, \nu)} \cdot x = (\mathfrak{g}_\mu \oplus \mathfrak{k}_\nu) \cdot x = (\mathfrak{g}_\mu \cdot x + \mathfrak{k}_\nu \cdot x)$, so the symplectic normal space at x is $N_s(x) = \ker T_x J_{G \times K} / (\mathfrak{g}_\mu \cdot x + \mathfrak{k}_\nu \cdot x)$. The claims about the quotient action of K on $J_G^{-1}(\mu)/G_\mu$ are part of regular commuting reduction (Theorem 3.1), and in any case are easy to prove by “diagram-chasing”.

We will now show that $\theta : (G \times K)_x \rightarrow K_{[x]_{G_\mu}}, (g, k) \mapsto k$, is an isomorphism. To show it’s well-defined, let $(g, k) \in (G \times K)_x$, so $k \cdot x = g^{-1} \cdot x$. Since J_G is Ad^* -equivariant and $J_G^{-1}(\mu)$ is K -invariant, we have $\mu = J_G(x) = J_G((g, k) \cdot x) = g \cdot \mu$, so $g \in G_\mu$. This implies that $k \in K_{[x]_{G_\mu}}$. So θ is well-defined. It is clearly smooth, and a homomorphism. For every $k \in K_{[x]_{G_\mu}}$, we have $k \cdot x \in G_\mu x$; since G acts freely, there is a unique element $\gamma(k) \in G_\mu$ such that $k \cdot x = \gamma(k)^{-1} \cdot x$. Clearly $(\gamma(k), k) \cdot x = x$, so the map $k \mapsto (\gamma(k), k)$ is an inverse for θ . The smoothness of θ^{-1} is a consequence of the implicit function theorem applied to the restricted action $F : G_\mu \times K_{[x]_{G_\mu}} \rightarrow G_\mu \cdot x$ given by $F(g, k) = (g, k) \cdot x$. Indeed, note that $(G \times K)_x = F^{-1}(x)$, and that $D_1 F(g, k)$ is surjective for every $(g, k) \in G_\mu \times K_{[x]_{G_\mu}}$, since the G action is free. Hence θ is a Lie group isomorphism. We identify $(G \times K)_x$ with $K_{[x]_{G_\mu}}$ via θ , calling both groups H .

Next, observe that

$$\begin{aligned} \ker T_x J_{G \times K} &= \ker T_x J_G \cap \ker T_x J_K = T_x J_G^{-1}(\mu) \cap \ker T_x J_K = \ker T_x (J_K|_{J_G^{-1}(\mu)}) \\ &= \ker T_x (J_{\bar{K}} \circ \pi_{G_\mu}) = (T_x \pi_{G_\mu})^{-1} (\ker T_{[x]_{G_\mu}} J_{\bar{K}}). \end{aligned}$$

Since $T_x \pi_{G_\mu}$ is surjective, this implies that $T_x \pi_{G_\mu}(\ker T_x J_{G \times K}) = \ker T_{[x]_{G_\mu}} J_{\bar{K}}$. The map $T\pi_{G_\mu}$ is a presymplectic submersion, by definition of the reduced symplectic form on

$J_G^{-1}(\mu)/G_\mu$. Hence Lemma 3.4 implies that $\overline{T_x \pi_{G_\mu}}$, as defined in the statement of the theorem, is a well-defined symplectic isomorphism from $N_s(x)$ to $N_s([x]_{G_\mu})$.

The projection π_{G_μ} is K -equivariant, by definition of the quotient action of K . Since we have already shown that $(g, k) \in (G \times K)_x$ implies $g \in G_\mu$, the H -equivariance of π_{G_μ} is easily checked. The H -equivariance of $T_x \pi_{G_\mu}$, and hence $\overline{T_x \pi_{G_\mu}}$, follows. ■

4 The symplectic normal space of a cotangent-lifted action

The main result of the section will be a characterisation of the symplectic normal space N_s to the orbit of a cotangent-lifted action, given in Theorem 4.9. In two special cases this leads to new splittings of N_s , given in Corollaries 4.10 and 4.13. Our analysis of the special case $G_q \subset G_\mu$ and much of the general set-up developed in this section will be used later in the cotangent bundle slice theorem (Theorem 5.6). We also note implications for singular reduction, in Theorem 4.8 and Remark 4.15, and the reconstruction equations (bundle equations), in Equations 18 and 19.

Let G act properly by cotangent lifts on T^*Q , with momentum map J , and let $z \in T_q^*Q$ and $\mu = J(z)$. Let $K = G_q$ and $H = G_z$, and let $\mathfrak{g}, \mathfrak{g}_\mu, \mathfrak{k}, \mathfrak{h}$ be the Lie algebras of G, G_μ, K and H .

Lemma 4.1 (i) $H \subset K$ (ii) $H \subset G_\mu$ (iii) $\mathfrak{k} \subset \ker \mu$ (iv) If K is normal in G , then $K \subset G_\mu$.

Proof. Claim (i) is clear from $z \in T_q^*Q$; (ii) follows from the equivariance of J . (iii) The definition of J gives $\langle \mu, \xi \rangle = \langle z, \xi_Q(q) \rangle = 0$ for all $\xi \in \mathfrak{k}$. (iv) For every $g \in G$ and $k \in K$ we have $gkg^{-1}k^{-1} \in K$. Differentiating with respect to g gives $\xi - \text{Ad}_{k^{-1}}\xi \in \mathfrak{k}$. Thus, for every $k \in K$ and $\xi \in \mathfrak{g}$, we have $\langle \text{Ad}_{k^{-1}}^* \mu - \mu, \xi \rangle = \langle \mu, \text{Ad}_{k^{-1}}\xi - \xi \rangle = 0$, in other words $k \in G_\mu$. ■

There exist simple examples in which H is a proper subset of $K \cap G_\mu$. The complex relationship between the different isotropy subgroups is one of the key difficulties of the subject.

We begin by applying Palais' slice theorem (Theorem 2.4) to the configuration space Q . Choose a K -invariant Riemannian metric on some neighbourhood of q in Q , and let A be the orthogonal complement to $\mathfrak{g} \cdot q$ in $T_q Q$, written $A = (\mathfrak{g} \cdot q)^\perp$. By Palais' slice theorem (Theorem 2.4), there exists a K -invariant neighbourhood U of 0 in A such that the map

$$s : G \times_K U \longrightarrow Q \tag{8}$$

$$[g, a]_K \longmapsto g \cdot \exp_q a \tag{9}$$

is a G -equivariant embedding. The cotangent lift

$$T^*s^{-1} : T^*(G \times_K U) \rightarrow T^*Q \tag{10}$$

is a G -equivariant symplectic embedding onto a neighbourhood of z (symplectic with respect to the standard cotangent bundle symplectic forms).

We next make the following key observation (explained fully in Proposition 4.4):

$T^*(G \times_K U)$ is a reduced space for the lifted twist action of K on $T^*(G \times U)$.

We will apply commuting reduction to $T^*(G \times U)$, with the second action being the lift of left multiplication by G . We first fill in the details of the passage to $T^*(G \times U)$, and state some basic facts for later use.

Let N be any manifold on which K acts (we have in mind $N = U$ or $N = A$, but the following facts are general). Recall from the Equation 5 the following two left actions on $G \times N$, which commute and are both free and proper:

$$\begin{aligned} K \text{ acts by } \textit{twisting} : k^K \cdot (g, n) &= (gk^{-1}, k \cdot n) \\ G \text{ acts by } \textit{left multiplication} : h^G \cdot (g, n) &= (hg, n). \end{aligned} \quad (11)$$

Note that, since K is a subset of G , there is room for confusion of the two actions, so we have introduced superscripts to identify them. Each of these actions has a corresponding tangent-lifted action on $T(G \times N) \cong TG \times TN$ and cotangent-lifted action on $T^*(G \times N) \cong T^*G \times T^*N$. It is easy to see that these actions commute and are free and proper.

Throughout this article, we will identify TG with $G \times \mathfrak{g}$ and T^*G with $G \times \mathfrak{g}^*$ by left trivialisation,

$$\begin{aligned} TG &\xrightarrow{\cong} G \times \mathfrak{g} \quad \text{and} \quad T^*G \xrightarrow{\cong} G \times \mathfrak{g}^* \\ TL_g(\xi) &\longmapsto (g, \xi) \quad T^*L_{g^{-1}}(\nu) \longmapsto (g, \nu) \end{aligned}$$

where L_g is left multiplication by g . The following basic properties of the left and right multiplication actions are well known.

Lemma 4.2 *Let G be a Lie group. With respect to the left trivialisations of TG and T^*G , the left and right multiplication actions of G on itself have the following lifted actions and infinitesimal lifted actions:*

$$\begin{aligned} \text{tangent: } h^L \cdot (g, \xi) &= (hg, \xi) & h^R \cdot (g, \xi) &= (gh^{-1}, \text{Ad}_h \xi) \\ \text{cotangent: } h^L \cdot (g, \nu) &= (hg, \nu) & h^R \cdot (g, \nu) &= (gh^{-1}, \text{Ad}_h^* \nu) \\ \text{infinitesimal tangent: } \eta^L \cdot (g, \xi) &= (\text{Ad}_{g^{-1}} \eta, 0) & \eta^R \cdot (g, \xi) &= (-\eta, \text{Ad}_\eta \xi) \\ \text{infinitesimal cotangent: } \eta^L \cdot (g, \nu) &= (\text{Ad}_{g^{-1}} \eta, 0) & \eta^R \cdot (g, \nu) &= (-\eta, -\text{Ad}_\eta^* \nu) \end{aligned}$$

The cotangent-lifted actions have the following momentum maps, with respect to the canonical symplectic form on T^*G :

$$J_L(g, \nu) = \text{Ad}_{g^{-1}}^* \nu, \quad J_R(g, \nu) = -\nu.$$

The momentum map J_L is invariant under the right multiplication action, and J_R is invariant under the left multiplication action.

There are obvious corresponding properties for the G and K actions on $G \times N$. In particular, we have the following:

Remark 4.3 Let G and K act on $G \times N$ as in Equation 11. Then the momentum maps for the cotangent-lifted actions on $T^*(G \times N) \cong G \times \mathfrak{g}^* \times T^*N$, with respect to the canonical symplectic form on $T^*(G \times N)$, are

$$J_G(g, \nu, w) = \text{Ad}_{g^{-1}}^* \nu, \quad J_K(g, \nu, w) = -\nu|_{\mathfrak{k}} + J_N(w),$$

where $J_N(w)$ is the momentum map for the cotangent-lifted action of K on T^*N . The previous lemma implies that J_G is invariant under the twist action of K and J_K is invariant under the left multiplication action of G . If N is a vector space, we can identify TN with $N \times N$ and T^*N with $N \times N^*$, so

$$T(G \times N) \cong G \times \mathfrak{g} \times N \times N \quad \text{and} \quad T^*(G \times N) \cong G \times \mathfrak{g}^* \times N \times N^*,$$

where the first and third components are the base space, and the second and fourth are the (co-)tangent fibers. These identifications will be used throughout this article. In these coordinates, and using the diamond notation (see Equation 2), $J_K(g, \nu, a, \delta) = -\nu|_{\mathfrak{k}} + a \diamond_{\mathfrak{k}} \delta$.

We are now in a position to apply reduction theorems to the two actions on $T^*(G \times N)$. We begin by studying the reduced space at 0 for the K action, using cotangent bundle reduction (Theorem 2.3). Note that $(G \times N)/K = G \times_K N$. The map φ in Theorem 2.3 takes the following form:

$$\varphi : (J_K^{-1}(0) \subset T^*(G \times N)) \rightarrow T^*(G \times_K N), \quad \langle \varphi(p), T\pi_K(v) \rangle = \langle p, v \rangle, \quad (12)$$

where $\pi_K : G \times N \rightarrow G \times_K N$ is projection. Recall that G has a quotient action on $G \times_K N$, and so G acts on $T^*(G \times_K N)$ by cotangent lifts. The projection π_K is G -equivariant by definition of the G action on $G \times_K A$, so $T\pi_K$ is G -equivariant with respect to the tangent lifted actions, from which it follows that φ is G -equivariant. Since $J_K^{-1}(0)$ is G -invariant, the G action descends to one on $J_K^{-1}(0)/K$. It is easily verified that this quotient action is symplectic; in fact this claim is part of Theorem 3.1 (regular commuting reduction). Applying Theorems 2.3 and 3.1 gives the following result.

Proposition 4.4 Let G and K act on $T^*(G \times N)$ as above, with momentum maps J_K and J_G respectively. Let φ be defined as in Equation 12. Then φ is a G -equivariant K -invariant surjective submersion that descends to a G -equivariant symplectomorphism

$$\bar{\varphi} : J_K^{-1}(0)/K \rightarrow T^*(G \times_K N),$$

with respect to the reduced symplectic form on $J_K^{-1}(0)/K$ and the canonical symplectic form on $T^*(G \times_K N)$. If J' is the momentum map for the G action on $T^*(G \times_K N)$, then the restriction of J_G to $J_K^{-1}(0)$ equals $J' \circ \varphi$.

Taking $N = A$ and combining this result with the map T^*s^{-1} from Equation 10, we have a G -equivariant symplectic embedding,

$$((J_K^{-1}(0) \cap (G \times \mathfrak{g}^* \times U \times A^*))/K) \xrightarrow{\bar{\varphi}} T^*(G \times_K U) \xrightarrow{T^*s^{-1}} T^*Q \quad (13)$$

In particular, there exists an $x \in J_K^{-1}(0)$ such that $T^*s^{-1}(\varphi(x)) = z \in T_q^*Q$. Since $s([e, 0]_K) = q$, we see that $\varphi(x)$ has base point $[e, 0]_K$. Since φ covers $\pi_K : G \times U \rightarrow G \times_K U$ and is K -invariant, we can choose x to have base point $(e, 0)$; in fact, since φ is injective, this uniquely determines x . So $x = (e, \nu, 0, \alpha)$, for some $\nu \in \mathfrak{g}^*$ and some $\alpha \in A^*$. Using Proposition 4.4, we have $\nu = J_G(x) = J'(\varphi(x)) = J(z) = \mu$. We can also show that $\alpha = z|_A$. Indeed, for every $v \in A$, we have

$$\begin{aligned} \langle \alpha, v \rangle &= \langle \varphi(e, \mu, 0, \alpha), T\pi_K(e, 0, 0, v) \rangle = \langle T^*s(z), T\pi_K(e, 0, 0, v) \rangle \\ &= \langle z, T(s \circ \pi_K)(e, 0, 0, v) \rangle = \langle z, v \rangle, \end{aligned}$$

where in the last line we have used $s \circ \pi_K(g, a) = g \cdot \exp_z a$, and the fact that the derivative at 0 of \exp_z is the identity. In summary, we have shown:

Lemma 4.5 *Let $\alpha = z|_A$ and $x = (e, \mu, 0, \alpha)$. Then $T^*s^{-1}(\varphi(x)) = z$.*

Remark 4.6 *Recall that $H = G_z$ and note that $G_z = G_{[x]_K}$, where $[x]_K = \pi_K(x)$. By definition of the G and K actions on $T^*(G \times A)$, we have $H = G_\mu \cap K_\alpha$. We note for later use that if $H = K$ then $K_\alpha = K$; while if $G_\mu = G$ then $H = K_\alpha$.*

Applying singular commuting reduction (Theorem 3.2) gives the following picture:

$$\begin{array}{ccc} & T^*(G \times A) & \\ \text{reduction by } G \swarrow & & \searrow \text{reduction by } K \\ ? \cong J_G^{-1}(\mu)/G_\mu & & J_K^{-1}(0)/K \stackrel{\text{local}}{\cong} T^*Q \\ \text{reduction by } K \searrow & & \swarrow \text{reduction by } G \\ & ? \stackrel{\text{local}}{\cong} J^{-1}(\mu)/G_\mu & \end{array}$$

We now compute $J_G^{-1}(\mu)/G_\mu$. Note that the G action leaves A untouched, so

$$J_G^{-1}(\mu)/G_\mu \cong (J_L^{-1}(\mu)/G_\mu) \times T^*A,$$

where $J_L^{-1}(\mu)/G_\mu$ is the symplectic reduced space at μ for the lifted left multiplication action of G on T^*G . It is well-known that $J_L^{-1}(\mu)/G_\mu$ is symplectomorphic to \mathcal{O}_μ with the KKS symplectic form $\omega_{\mathcal{O}_\mu}^-$ (defined in Eq. 1); see for example, Appendix B.4 of [CB97]. The isomorphism is $[g, \nu]_{G_\mu} \mapsto \nu$ (using the left trivialisation of T^*G .) We have almost proven the following proposition; the remain claims in it are easily verified.

Proposition 4.7 *The map θ defined by*

$$\begin{aligned} \theta : J_G^{-1}(\mu) &\rightarrow \mathcal{O}_\mu \times T^*A \\ (g, \nu, a, \delta) &\longmapsto (\nu, a, \delta) \end{aligned} \tag{14}$$

is a surjective submersion that descends to diffeomorphism

$$\begin{aligned}\bar{\theta} : J_G^{-1}(\mu) / G_\mu &\longrightarrow \mathcal{O}_\mu \times T^*A \\ [g, \nu, a, \delta]_{G_\mu} &\longmapsto (\nu, a, \delta)\end{aligned}$$

that is symplectic with respect to the reduced symplectic form on the left and $\omega_{\mathcal{O}_\mu}^- + \omega_{T^*A}$ on the right. The pushed-forward K action is $k \cdot (\nu, a, \delta) = (\text{Ad}_{k^{-1}}^* \nu, k \cdot a, k \cdot \delta)$. It has momentum map $J'_K(\nu, a, \alpha) = -\nu|_{\mathfrak{k}} + a \diamond_{\mathfrak{k}} \alpha$.

Our results so far, combined with singular commutation reduction (Theorem 3.2), give the following normal form for reduced spaces for cotangent-lifted actions.

Theorem 4.8 *Let G act properly on a manifold Q and by cotangent lifts on T^*Q with momentum map J . Let $z \in T_q^*Q$ and $K = G_q$. Let $\mu = J(z)$ and suppose that G_μ is compact and \mathcal{O}_μ is locally closed. Let A be a K -invariant complement to $\mathfrak{g} \cdot q$ with respect to some K -invariant metric. Then there is a local Poisson diffeomorphism between $J^{-1}(\mu) / G_\mu$ and the reduced space at 0 for the product action of K on the space $\mathcal{O}_\mu \times T^*A$ with symplectic form $\omega_{\mathcal{O}_\mu}^- + \omega_{T^*A}$, where K has the coadjoint action on \mathcal{O}_μ and the cotangent lifted action on T^*A .*

Our main aim in this section is to characterise the symplectic normal space $N_s(z)$. To this end, we apply tangent-level commuting reduction (Theorem 3.5) to the actions of G and K on $T^*(G \times A)$. Recall from Lemma 4.5 and Remark 4.6 that $x = (e, \mu, 0, \alpha)$ and $H = G_z = G_{[x]_K} = G_\mu \cap K_\alpha$. It is easy to check that $(G \times K)_x = \{(h, h) \mid h \in H\}$, and that $H = K_{[x]_{G_\mu}}$ as well. A generalisation of this observation appears in Theorem 3.5. As in that theorem, we will identify $(G \times K)_x$ with H . The subgroup H acts on all three of the symplectic normal spaces $N_s([x]_{G_\mu})$, $N_s(x)$ and $N_s([x]_K)$ in the usual way. Theorem 3.5 (tangent-level commuting reduction) implies that the following maps are H -equivariant vector space symplectomorphisms,

$$N_s([x]_{G_\mu}) \xrightarrow[\cong]{\overline{T_x \pi_{G_\mu}}} N_s(x) \xrightarrow[\cong]{\overline{T_x \pi_K}} N_s([x]_K), \quad (15)$$

where the overbars denote the quotient maps, as in Remark 3.4.

Now recall from Equation 13 that $T^*s^{-1} \circ \bar{\varphi}$ is a G -equivariant symplectomorphism from a neighbourhood of $[x]_K$ to a neighbourhood of z . It follows that $\overline{T_{[x]_K}(T^*s^{-1} \circ \bar{\varphi})}$ is an H -equivariant symplectomorphism from $N_s([x]_K)$ to $N_s(z)$. Since $\varphi = \bar{\varphi} \circ \pi_K$, we can compose this with $\overline{T_x \pi_K}$ from above to give $\overline{T_x(T^*s^{-1} \circ \varphi)} : N_s(x) \rightarrow N_s(z)$. Similarly, Proposition 4.7 implies that the map $\overline{T_{[x]_{G_\mu}} \bar{\theta}} : N_s([x]_{G_\mu}) \rightarrow N_s(\mu, 0, \alpha)$ is an H -equivariant symplectomorphism; and we can compose this with $\overline{T_x \pi_{G_\mu}}$ to give $\overline{T_x \bar{\theta}} : N_s(x) \rightarrow N_s(z)$. Combining these results, we have the following,

Theorem 4.9 *In the above context (with s, φ and θ defined by Equations 8, 12 and 14, respectively), the composition $\overline{T_x \bar{\theta} \circ T_x(T^*s^{-1} \circ \varphi)^{-1}} : N_s(z) \rightarrow N_s(\mu, 0, \alpha)$ is an H -invariant symplectomorphism of symplectic normal spaces.*

The space $N_s(\mu, 0, \alpha)$ has simple forms in the special cases $K \subset G_\mu$ and $\alpha = 0$. When $K \subset G_\mu$, the K action on \mathcal{O}_μ is trivial, so

$$N_s(\mu, 0, \alpha) \cong T_\mu \mathcal{O}_\mu \oplus N_s(0, \alpha), \quad (16)$$

the second summand being the symplectic normal space at $(0, \alpha)$ for the cotangent-lifted action of K on T^*A . Recall that the momentum map for the latter action is $J_A(a, \gamma) = a \diamond \gamma$. It follows that $dJ_A(0, \alpha)(b, \beta) = 0 \diamond \beta + b \diamond \alpha = b \diamond \alpha$, so $\ker dJ_A(0, \alpha) = (\mathfrak{k} \cdot \alpha)^\circ \oplus A^*$. Hence $N_s(0, \alpha) = \ker dJ_A(0, \alpha) / (\mathfrak{k} \cdot (0, \alpha)) \cong (\mathfrak{k} \cdot \alpha)^\circ \oplus (A^* / (\mathfrak{k} \cdot \alpha))$. It is not hard to show that the dual ι^* of the inclusion $\iota : (\mathfrak{k} \cdot \alpha)^\circ \hookrightarrow A$ descends to an isomorphism $\bar{\iota}^* : A^* / (\mathfrak{k} \cdot \alpha) \cong ((\mathfrak{k} \cdot \alpha)^\circ)^*$, and the map

$$N_s(0, \alpha) \cong (\mathfrak{k} \cdot \alpha)^\circ \oplus (A^* / (\mathfrak{k} \cdot \alpha)) \xrightarrow{(id, \bar{\iota}^*)} (\mathfrak{k} \cdot \alpha)^\circ \oplus ((\mathfrak{k} \cdot \alpha)^\circ)^* \cong T^*(\mathfrak{k} \cdot \alpha)^\circ \quad (17)$$

is a symplectomorphism and is H -equivariant with respect to the cotangent lift of the restriction of the K action on A to a H action on $(\mathfrak{k} \cdot \alpha)^\circ$. Thus we arrive at the following corollary to Theorem 4.9:

Corollary 4.10 *When $K \subset G_\mu$, there is an H -equivariant symplectomorphism*

$$N_s(z) \cong T_\mu \mathcal{O}_\mu \oplus T^*(\mathfrak{k} \cdot \alpha)^\circ.$$

Remark 4.11 *In light of Lemma 4.1 (iv), the above result applies whenever K is normal in G .*

Remark 4.12 *This corollary generalises a splitting established for free actions by Montgomery et al. (see [MMR84]).*

We now consider the case $\alpha = 0$. Recall that $\alpha = z|_A$, where $A = (\mathfrak{g} \cdot q)^\perp$; so, with respect to our choice of metric, this is the case where the conjugate momentum z is “purely in the group direction.” Since $J'_K(\nu, a, \alpha) = -\nu|_{\mathfrak{k}} + a \diamond \alpha$, it follows that $dJ'_K(\mu, 0, 0)(\rho, b, \beta) = -\rho|_{\mathfrak{k}}$. Note that this equals $dJ_\mu(\mu)(\rho)$, where J_μ is the momentum map for the coadjoint action of K on $(\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu}^-)$ namely $J_\mu(\nu) = -\nu|_{\mathfrak{k}}$. So $\ker dJ'_K(\mu, 0, 0) = \ker dJ_\mu(\mu) \oplus T^*A$. By Lemma 4.1 (iii), $J'_K(\mu) = J_\mu(\mu) = 0$, so $\mathfrak{k}_{J'_K(\mu)} = \mathfrak{k}_{J_\mu(\mu)} = \mathfrak{k}$. It follows that

$$N_s((\mu, 0, 0)) = (\ker dJ_\mu(\mu) \oplus T^*A) / (\mathfrak{k}_{J_\mu(\mu) \cdot \mu} \oplus \{(0, 0)\}) = N_s(\mu) \oplus T^*A$$

By the Reduction Lemma (or direct calculation), $N_s(\mu) = (\mathfrak{k} \cdot \mu)^{\omega^-} / (\mathfrak{k} \cdot \mu)$. Thus we have the following corollary to Theorem 4.9:

Corollary 4.13 *When $\alpha = 0$, the map in Theorem 4.9 is an H -equivariant symplectomorphism*

$$N_s(z) \cong N_s(\mu) \oplus T^*A = (\mathfrak{k} \cdot \mu)^{\omega^-} / (\mathfrak{k} \cdot \mu) \oplus T^*A,$$

where $N_s(\mu)$ is the symplectic normal space at μ for the coadjoint action of K on $(\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu}^-)$.

Remark 4.14 *The above corollary applies to all relative equilibria of simple mechanical systems. Indeed, if $z \in T_q^*Q$ is such a relative equilibrium then $z = \mathbb{F}L(\xi \cdot q)$ for some $\xi \in \mathfrak{g}$ (see [Mars92]). For any $v \in A$ we have $\langle z, v \rangle = \langle \xi \cdot q, v \rangle = 0$, since $A = (\mathfrak{g} \cdot q)^\perp$. Hence $\alpha = z|_A = 0$. More generally, the corollary applies to any point z such that the kernel of z includes some complement to $\mathfrak{g} \cdot q$, because we can choose our metric on Q such that this complement is $(\mathfrak{g} \cdot q)^\perp$.*

Remark 4.15 *Theorem 4.9 and its corollaries, when combined with Theorem 3.3, give cotangent-bundle-specific local models of symplectic reduced spaces. Indeed, if \mathcal{O}_μ is locally closed, there is a local Poisson diffeomorphism between the reduced space $J^{-1}(\mu)/G_\mu$ and the reduced space at 0 for the H action on $N_s(\mu, 0, \alpha)$. Note the similarity to Theorem 4.8, which shows that $J^{-1}(\mu)/G_\mu$ is isomorphic to the reduced space at 0 for the K action on $\mathcal{O}_\mu \times T^*A$. Thus, symplectic reduced spaces for cotangent bundles have two local models, corresponding to the two isotropy subgroups H and K . The model involving H and $N_s(\mu, 0, \alpha)$ is more “economical” in that H and $N_s(\mu, 0, \alpha)$ may be smaller than K and $\mathcal{O}_\mu \times T^*A$, but on the other hand the latter space is “simpler” and might be easier to work with in some situations.*

We end this section with the observation that Corollaries 4.10 and 4.13 lead to refinements of the so-called *reconstruction equations* or *bundle equations* [Ort98, OR02b, RWL02], which are a normal form for Hamilton’s equations in the coordinates given by the Hamiltonian Slice Theorem (Theorem 2.6). Consider the local symplectomorphism $G \times_H (\mathfrak{m}^* \times N_s) \rightarrow P$ given by the Hamiltonian Slice Theorem, for any proper globally Hamiltonian action of G on P , with $H = G_z$ as before; recall that \mathfrak{m} is an H -invariant complement to \mathfrak{h} in \mathfrak{g}_μ , where $\mu = J(z)$. A Hamiltonian on P pulls back to a Hamiltonian h on a neighbourhood of $[e, 0, 0]_H$ in $G \times_H (\mathfrak{m}^* \times N_s)$, with corresponding Hamiltonian vector field X_h . Using a local bundle chart around $[e, 0, 0]_H$ for the principal bundle $\pi_H : G \times \mathfrak{m}^* \times N_s \rightarrow G \times_H (\mathfrak{m}^* \times N_s)$, we can lift X_h to a smooth vector field on a neighbourhood of $(e, 0, 0)$ in $G \times \mathfrak{m}^* \times N_s$. This lift is not unique; however we can specify a unique lift by choosing an H -invariant complement \mathfrak{q} to \mathfrak{g}_μ , so that we now have $\mathfrak{g} = \mathfrak{q} + \mathfrak{m} + \mathfrak{h}$, and requiring that the component of the lifted vector field in the \mathfrak{h} direction be zero. The lifted vector field can now be written as $X = (TL_g(X_\mathfrak{m} + X_\mathfrak{q}), X_{\mathfrak{m}^*}, X_{N_s})$.

If it is possible to choose \mathfrak{q} to be G_μ -invariant then we say that μ is *split*. We assume this now for simplicity; the general case is considered in [RWL02]. Assuming μ is split, it can be shown that the lifted Hamiltonian vector field $X(g, \rho, v)$ is given by:

$$\begin{aligned} X_\mathfrak{q} &= 0 \\ X_\mathfrak{m} &= D_{\mathfrak{m}^*}(h \circ \pi)(\rho, v) \\ i_{X_{N_s}} \omega_{N_s} &= D_{N_s}(h \circ \pi)(\rho, v) \\ X_{\mathfrak{m}^*} &= \mathbb{P}_{\mathfrak{m}^*} \left(\text{ad}_{D_{\mathfrak{m}^*}(h \circ \pi)}^* \rho \right) + \text{ad}_{D_{\mathfrak{m}^*}(h \circ \pi)}^* J_{N_s}(v) \end{aligned}$$

Now suppose $P = T^*Q$ and $z \in T_q^*Q$. If $G_q \subset G_\mu$, we know from Corollary 4.10 that N_s is linearly symplectomorphic to $T_\mu \mathcal{O}_\mu \oplus T^*(\mathfrak{k} \cdot \alpha)^\circ$. Let $B = (\mathfrak{k} \cdot \alpha)^\circ$, so $N_s \cong T_\mu \mathcal{O}_\mu \oplus B \oplus B^*$.

The vector field X_{N_s} separates into three components X_μ, X_B and X_{B^*} and the equation $i_{X_{N_s}} \omega_{N_s} = D_{N_s} (h \circ \pi) (\rho, v)$ splits into the three equations,

$$\begin{aligned} i_{X_\mu} \omega_{\mathcal{O}_\mu^-} &= D_{T\mu\mathcal{O}_\mu} (h \circ \pi) (\rho, v) \\ X_B &= D_{B^*} (h \circ \pi) (\rho, v) \\ X_{B^*} &= -D_B (h \circ \pi) (\rho, v). \end{aligned} \tag{18}$$

(the last two equations being the canonical form for Hamilton's equations). The case $\alpha = 0$ in Corollary 4.13 is similar: the X_{N_s} equation is replaced by three equations,

$$\begin{aligned} i_{X_\mu} \omega_{\mathcal{O}_\mu^-, red} &= D_{N_s(\mu)} (h \circ \pi) (\rho, v) \\ X_A &= D_{A^*} (h \circ \pi) (\rho, v) \\ X_{A^*} &= -D_A (h \circ \pi) (\rho, v). \end{aligned} \tag{19}$$

5 A cotangent bundle slice theorem

In this section, we extend the Hamiltonian slice theorem (Theorem 2.6) in the case of a lifted action on a cotangent bundle. The main result is Theorem 5.6. We will consider only the case of fully isotropic momenta, $G_\mu = G$, for reasons that will be summarised in Remark 5.8. Our model for T^*Q will be $G \times_H (\mathfrak{m}^* \oplus N_s)$, as in the general Hamiltonian theorem, with the same symplectic form as in that theorem (definitions will be reiterated below). However, in contrast to the general Hamiltonian slice theorem, our isomorphism from the model space to T^*Q will be constructed explicitly, apart from the use of a Riemannian exponential in the base space Q . The construction will use the decomposition of N_s in Corollary 4.10.

As before, let G be a Lie group acting smoothly and properly by cotangent lifts on T^*Q , with momentum map J . Let $z \in T_q^*Q$ and $\mu = J(z)$, and let $K = G_q$ and $H = G_z$. We assume $G_\mu = G$. Let $\mathfrak{g}, \mathfrak{g}_\mu, \mathfrak{h}$ and \mathfrak{k} be the Lie algebras of G, G_μ, H and K respectively. Fix a K -invariant inner product on \mathfrak{g} and let $\mathfrak{m} = \mathfrak{k}^\perp$. Let N_s be the symplectic normal space at z . Our goal is to find a symplectic tube from $G \times_H (\mathfrak{m}^* \oplus N_s)$ to T^*Q that takes $[e, 0, 0]$ to z .

We first apply Palais' slice theorem in the configuration space Q . Fix a K -invariant Riemannian metric on Q and let $A = (\mathfrak{g} \cdot q)^\perp$. By the slice theorem, there exists a G -equivariant diffeomorphism $s : G \times_K V \rightarrow Q$ taking $[e, 0]_K$ to q , for some neighbourhood V of 0 in A . The cotangent lift of s is a G -equivariant symplectomorphism $T^*s^{-1} : T^*(G \times_K V) \rightarrow T^*Q$. Let $\varphi : J_K^{-1}(0) \rightarrow T^*(G \times_K A)$ be the cotangent bundle reduction map defined in Equation 12, and let $\alpha = z|_A$. Recall from Lemma 4.5 that $T^*s^{-1} \circ \varphi(e, \mu, 0, \alpha) = z$. Hence it will suffice to find a symplectic tube

$$\begin{aligned} \tau : G \times_H (\mathfrak{m}^* \oplus N_s) &\longrightarrow T^*(G \times_K A) \\ [e, 0, 0]_H &\longmapsto \varphi(e, \mu, 0, \alpha). \end{aligned} \tag{20}$$

Since $G_\mu = G$, Corollary 4.10 says that the symplectic normal space N_s is H -equivariantly symplectomorphic to $T^*B \cong B \oplus B^*$, where $B = (\mathfrak{k} \cdot \alpha)^\circ \subset A$; the symplectic form on T^*B

is the canonical one, and the H action on T^*B is the cotangent-lift of the restriction to H and B of the K action on A . We will identify N_s with $B \oplus B^*$. Recall that the presymplectic form on $G \times_H (\mathfrak{m}^* \oplus N_s)$ in the Hamiltonian slice theorem is defined using a symplectic form $\Omega_Z = \Omega_c + \Omega_\mu + \Omega_{N_s}$ on $Z = G \times \mathfrak{g}_\mu^* \times N_s$ (see Equation 7). Since we are assuming $G_\mu = G$, we have $Z = G \times \mathfrak{g}^* \times B \times B^*$, which we identify with $T^*(G \times B)$ by left trivialisation of T^*G . The twist action of H on Z becomes the cotangent lift of the twist action of H on $G \times B$. The form Ω_μ is a pull-back of a symplectic form on \mathcal{O}_μ , which is trivial in this case, so $\Omega_Z = \Omega_c + \Omega_{N_s}$. Since Ω_c is the pull-back by left-trivialisation of the canonical symplectic form on T^*G , and Ω_{N_s} is the canonical symplectic form on T^*B , the identification of Z with $T^*(G \times B)$ makes Ω_Z the canonical symplectic form on $T^*(G \times B)$. Note that, unlike in the general case, this Ω_Z is nondegenerate everywhere.

The symplectic form on $G \times_H (\mathfrak{m}^* \oplus N_s)$ is defined via an isomorphism with $J_H^{-1}(0)/H$, where J_H is the momentum map of the H action on $Z = T^*(G \times B)$. The isomorphism, defined earlier in Equations 6 and 7, is

$$\begin{aligned} L : G \times_H (\mathfrak{m}^* \oplus B \oplus B^*) &\longrightarrow J_H^{-1}(0)/H \\ [g, \nu, a, \delta]_H &\longmapsto [g, \nu + a \diamond_{\mathfrak{h}} \delta, a, \delta]_H \end{aligned}$$

The symplectic form on $G \times_H (\mathfrak{m}^* \oplus B \oplus B^*)$ is defined as the pull-back by L of the reduced symplectic form on $J_H^{-1}(0)/H$. Since L is clearly G -equivariant, it is a symplectic tube.

In the present case, cotangent bundle reduction (Theorem 2.3) shows that $J_H^{-1}(0)/H$ is isomorphic to $T^*(G \times_H B)$. Let ψ and $\bar{\psi}$ be the maps in the cotangent bundle reduction theorem,

$$\begin{array}{ccc} J_H^{-1}(0) & \hookrightarrow & T^*(G \times B) \\ \pi_{Z,H} \downarrow & \searrow \psi & \\ J_H^{-1}(0)/H & \xrightarrow{\bar{\psi}} & T^*(G \times_H B) \end{array}$$

It is easily checked that $\bar{\psi}$ is G -equivariant, by the same reasoning as used in Proposition 4.4. Note that

$$(\bar{\psi} \circ L)([g, \nu, a, \delta]_H) = \psi(g, \nu + a \diamond_{\mathfrak{h}} \delta, a, \delta). \quad (21)$$

In particular, $(\bar{\psi} \circ L)([e, 0, 0, 0]_H) = \psi(e, 0, 0, 0)$. Thus, to find a tube τ as in Equation 20, it suffices to find a G -equivariant symplectomorphism

$$\begin{aligned} \bar{\sigma} : T^*(G \times_H B) &\longrightarrow T^*(G \times_K A) \\ \psi(e, 0, 0, 0) &\longmapsto \varphi(e, \mu, 0, \alpha) \end{aligned} \quad (22)$$

(we will have to restrict the domain of $\bar{\sigma}$ in the general case).

The $z = 0$ case. In the simplest case, $z = 0 \in T_q^*Q$, we have $\mu = \alpha = 0$, $H = K$, $B = A$ and $\psi = \varphi$, so $\bar{\sigma}$ may be chosen to be the identity map on $T^*(G \times_K A)$. Composing this with the maps $\bar{\psi} \circ L$ and T^*s^{-1} gives the symplectic tube

$$\begin{aligned} G \times_K (\mathfrak{m}^* \oplus V \oplus A^*) &\longrightarrow T^*Q \\ [g, \nu, a, \delta]_K &\longmapsto T^*s^{-1} \circ \varphi(g, \nu + a \diamond_{\mathfrak{h}} \delta, a, \delta), \end{aligned}$$

where V is the neighbourhood of 0 in A given by Palais' slice theorem applied at $q \in Q$.

In general, μ and α may be nonzero. However, if $H = K$, we may take $\bar{\sigma}$ to be a simple shift map, as in the following lemma. Note that, since $H = K_\alpha$ and $B = (\mathfrak{k} \cdot \alpha)^\circ$, the condition $H = K$ is equivalent to $B = A$.

Lemma 5.1 *If $G_\mu = G$ then the shift map $\Sigma_{(\mu, \alpha)} : (g, \nu, a, \delta) \mapsto (g, \mu + \nu, a, \alpha + \delta)$, from $T^*(G \times A)$ to itself, is symplectic and G -equivariant. If $H = K$ then $B = A$ and $\Sigma_{(\mu, \alpha)}$ leaves J_H invariant and is H -equivariant. The “quotient” of $\Sigma_{(\mu, \alpha)}$ by ψ , the map*

$$\begin{aligned} \bar{\Sigma}_{(\mu, \alpha)} : T^*(G \times_H B) &\longrightarrow T^*(G \times_H B) \\ \psi(g, \nu, a, \delta) &\longmapsto \psi(g, \mu + \nu, a, \alpha + \delta) \end{aligned}$$

is a G -equivariant symplectomorphism.

Proof. It is clear from the local coordinate formula $dq^i \wedge dp_i$ that a canonical cotangent bundle symplectic form is invariant under shifts in the p variable. The G -equivariance is also clear. Now suppose $H = K$, which implies $B = A$, as explained above. Recall that $J_H(g, \nu, a, \delta) = -\nu|_{\mathfrak{h}} + a \diamond_{\mathfrak{h}} \delta$. Since $\mathfrak{k} \subset \ker \mu$ (see Lemma 4.1 (iii)), it follows that $-\mu|_{\mathfrak{h}} = 0$. From $H = K_\alpha$ (see Remark 4.6) it follows that $a \diamond_{\mathfrak{h}} \alpha = 0$. Hence J_H is invariant under $\Sigma_{(\mu, \alpha)}$. The H -equivariance follows from the linearity of the H actions on \mathfrak{g}^* and B^* together with $H \subset G_\mu \cap K_\alpha$.

Since $\Sigma_{(\mu, \alpha)}$ is a G - and H -equivariant symplectomorphism leaving $J_H^{-1}(0)$ invariant, it descends to a G -equivariant symplectomorphism from $J_H^{-1}(0)/H$ to itself. This map induces $\bar{\Sigma}_{(\mu, \alpha)}$ via the G -equivariant symplectomorphism $\bar{\varphi} : J_H^{-1}(0)/H \rightarrow T^*(G \times_H B)$. ■

The case $H = K$. Subcases include: $z = 0$; $\alpha = 0$; and all relative equilibria of simple mechanical systems (see Remark 4.14) (recall that we are assuming $G_\mu = G$ throughout this section). When $H = K$ we have $B = A$, so $T^*(G \times_H B) = T^*(G \times_K A)$ and $\psi = \varphi$. Thus we may take $\bar{\sigma} = \bar{\Sigma}_{(\mu, \alpha)}$. Composing this with the maps $\bar{\psi} \circ L$ and T^*s^{-1} gives

$$\begin{aligned} G \times_K (\mathfrak{m}^* \oplus V \oplus A^*) &\longrightarrow T^*Q \\ [g, \nu, a, \delta]_K &\longmapsto T^*s^{-1} \circ \varphi(g, \mu + \nu + b \diamond_{\mathfrak{h}} \delta, b, \alpha + \delta), \end{aligned}$$

where V is the neighbourhood of 0 in A given by Palais' slice theorem applied at $q \in Q$.

The general case is more difficult. We identify B^* with $(\mathfrak{k} \cdot \alpha)^\perp \subset A^*$. It is easily checked that the shift formula $(g, \nu, a, \delta) \mapsto (g, \mu + \nu, a, \alpha + \delta)$, as a map from $T^*(G \times B)$ to $T^*(G \times A)$, need *not* map $J_H^{-1}(0)$ into $J_K^{-1}(0)$, so cannot be used directly to define a map $\bar{\sigma}$ as in Equation 22. We will look for a map as close as possible to this shift map but with image contained in $J_K^{-1}(0)$. We will conclude in Lemma 5.3 that there is a unique map of the form $(g, \nu, a, \delta) \mapsto (g, \mu + \nu, a + c, \alpha + \delta)$, for $c \in B^\perp$, that accomplishes this.

We proceed by characterising the space $(G \times \mathfrak{g}^* \times A \times (\alpha + B^*)) \cap J_K^{-1}(0)$. We decompose A as $\left((\mathfrak{k} \cdot \alpha)^\perp\right)^\circ \oplus (\mathfrak{k} \cdot \alpha)^\circ = B^\perp \oplus B$; this splitting is H -invariant, since H fixes α (see Remark 4.6). By definition of \mathfrak{m} , we have an H -equivariant splitting $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ (recall that we are assuming $G_\mu = G$). It is easily checked that \mathfrak{k} splits H -equivariantly as $\mathfrak{k} = (\mathfrak{m} \cap \mathfrak{k}) \oplus \mathfrak{h}$.

Lemma 5.2

$$(G \times \mathfrak{g}^* \times A \times (\alpha + B^*)) \cap J_K^{-1}(0) \\ = \left\{ (g, \nu, b + c, \alpha + \delta) \mid b \in B, c \in B^\perp, J_H(g, \nu, b, \delta) = 0 \text{ and } (-\nu + b \diamond \delta + c \diamond (\alpha + \delta))|_{\mathfrak{m} \cap \mathfrak{k}} = 0 \right\}$$

and this is a submanifold of $T^*(G \times A)$.

Proof. Recall that $J_K(g, \nu, a, \beta) = -\nu|_{\mathfrak{k}} + a \diamond_{\mathfrak{k}} \beta$. The restriction of J_K to $G \times \mathfrak{g}^* \times A \times (\alpha + B^*)$ is a submersion, since $\nu \mapsto \nu|_{\mathfrak{k}}$ is one. It follows that $(G \times \mathfrak{g}^* \times A \times (\alpha + B^*)) \cap J_K^{-1}(0)$ is a submanifold of $(G \times \mathfrak{g}^* \times A \times (\alpha + B^*))$, and hence of $G \times \mathfrak{g}^* \times A \times A^*$.

Now let $(g, \nu, b + c, \alpha + \delta) \in (G \times \mathfrak{g}^* \oplus A \oplus (\alpha + B^*))$, with $b \in B$ and $c \in B^\perp$. Then $J_K(g, \nu, b + c, \alpha + \delta) = -\nu|_{\mathfrak{k}} + (b + c) \diamond_{\mathfrak{k}} (\alpha + \delta)$. Since $b \in B = (\mathfrak{k} \cdot \alpha)^\circ$, it follows that $b \diamond_{\mathfrak{k}} \alpha = 0$. For any $\xi \in \mathfrak{h}$ we have $\xi \cdot \alpha = 0$ and $\xi \cdot \delta \in (\mathfrak{k} \cdot \alpha)^\perp$, and so $\langle c, \xi \cdot (\alpha + \delta) \rangle = 0$; it follows that $c \diamond_{\mathfrak{h}} (\alpha + \delta) = 0$. Thus

$$\begin{aligned} (g, \nu, b + c, \alpha + \delta) \in J_K^{-1}(0) &\iff \nu|_{\mathfrak{k}} = b \diamond_{\mathfrak{k}} \delta + c \diamond_{\mathfrak{m} \cap \mathfrak{k}} (\alpha + \delta) \\ &\iff \nu|_{\mathfrak{h}} - b \diamond_{\mathfrak{h}} \delta = -\nu|_{\mathfrak{m} \cap \mathfrak{k}} + b \diamond_{\mathfrak{m} \cap \mathfrak{k}} \delta + c \diamond_{\mathfrak{m} \cap \mathfrak{k}} (\alpha + \delta) \\ &\iff \nu|_{\mathfrak{h}} - b \diamond_{\mathfrak{h}} \delta = 0 \text{ and } -\nu|_{\mathfrak{m} \cap \mathfrak{k}} + b \diamond_{\mathfrak{m} \cap \mathfrak{k}} \delta + c \diamond_{\mathfrak{m} \cap \mathfrak{k}} (\alpha + \delta) = 0 \\ &\iff J_H(g, \nu, b, \delta) = 0 \text{ and } (-\nu + b \diamond \delta + c \diamond (\alpha + \delta))|_{\mathfrak{m} \cap \mathfrak{k}} = 0. \end{aligned}$$

■

Lemma 5.3 *Let U be an H -invariant neighbourhood of 0 in B^* such that the map*

$$\begin{aligned} t : K \times_H U &\longrightarrow A^* \\ [k, \delta]_H &\longmapsto k \cdot (\alpha + \delta) \end{aligned}$$

is injective; such a U always exists. Then

1. *For every $\delta \in U$, the map $\Gamma_\delta^* : (\mathfrak{m} \cap \mathfrak{k})^* \longmapsto B^\perp$ defined by*

$$\langle \Gamma_\delta^*(\nu), \xi \cdot (\alpha + \delta) + \varepsilon \rangle = \langle \nu, \xi \rangle, \quad (23)$$

for every $\xi \in \mathfrak{m} \cap \mathfrak{k}$ and $\varepsilon \in B^$, is H -equivariant and has an inverse given by*

$$c \longmapsto -c \diamond_{\mathfrak{m} \cap \mathfrak{k}} (\alpha + \delta).$$

2. *The map σ defined by*

$$\begin{aligned} \sigma : (G \times \mathfrak{g}^* \times B \times U) \cap J_H^{-1}(0) &\longrightarrow (G \times \mathfrak{g}^* \times A \times (\alpha + U)) \cap J_K^{-1}(0) \\ (g, \nu, a, \delta) &\longmapsto (g, \mu + \nu, a + \Gamma_\delta^*(-\nu|_{\mathfrak{m} \cap \mathfrak{k}} + a \diamond_{\mathfrak{m} \cap \mathfrak{k}} \delta), \alpha + \delta) \end{aligned}$$

is the unique function, with this domain and range, of the form $(g, \nu, a, \delta) \longmapsto (g, \mu + \nu, a + c, \alpha + \delta)$ for $c \in B^\perp$. It is a presymplectic diffeomorphism, with respect to the canonical symplectic forms on $T^(G \times B)$ and $T^*(G \times A)$. It is equivariant*

with respect to both the left multiplication action by G and the twist action by H . It descends to a G -equivariant symplectic embedding $\bar{\sigma}$ defined by the following commutative diagram, where $W = (G \times \mathfrak{g}^* \times B \times U) \cap J_H^{-1}(0)$,

$$\begin{array}{ccc} (W \subset J_H^{-1}(0)) & \xrightarrow{\sigma} & J_K^{-1}(0) \\ \downarrow \psi & & \downarrow \varphi \\ (\psi(W) \subset T^*(G \times_H B)) & \xrightarrow{\bar{\sigma}} & T^*(G \times_K A) . \end{array}$$

The image of $\bar{\sigma}$ is an open subset of $T^*(G \times_K A)$.

Proof. (1) Note that $H = K_\alpha$ and we are identifying B^* with $(\mathfrak{k} \cdot \alpha)^\perp$, which is a linear slice for the K action on A^* . The slice theorem for linear actions (Theorem 2.5) shows the existence of an H -invariant neighbourhood U of 0 in B^* such that the map t above is injective and that, given any such U the map t is a K -equivariant diffeomorphism onto a K -invariant neighbourhood of α . Let $\pi_H : K \times U \rightarrow K \times_H U$ be projection. The composition $t \circ \pi_H$ is a submersion. For any $k \in K$ and $\delta \in U$, the kernel of $T_{(k,\delta)}\pi_H$ is $\{(-\zeta, \zeta \cdot \delta) \in \mathfrak{k} \oplus B^* \mid \zeta \in \mathfrak{h}\}$, which is a complement to the space $(\mathfrak{m} \cap \mathfrak{k}) \oplus B^*$ in $\mathfrak{k} \oplus B^*$, so

$$\begin{aligned} T_{(e,\delta)}(t \circ \pi_H)|_{(\mathfrak{m} \cap \mathfrak{k}) \oplus B^*} : (\mathfrak{m} \cap \mathfrak{k}) \oplus B^* &\longrightarrow T_{(\alpha+\delta)}A^* \cong A^* \\ (\xi, \varepsilon) &\longmapsto \xi \cdot (\alpha + \delta) + \varepsilon \end{aligned}$$

is an isomorphism. It follows that Equation 23 defines a map Γ_δ^* from $(\mathfrak{m} \cap \mathfrak{k})^*$ to A ; its image is clearly contained in $(B^*)^\circ \cong B^\perp$.

It is easily checked that Γ_δ^* is H -equivariant and has the stated inverse.

(2) We first check that σ is well-defined; the only part that needs checking is that its image is contained in the target space. It follows from Claim 1 that the condition $(-\nu + b \diamond \delta + c \diamond (\alpha + \delta))|_{\mathfrak{m} \cap \mathfrak{k}} = 0$ in Lemma 5.2 is equivalent to $c = \Gamma_\delta^*((-\nu + b \diamond \delta)|_{\mathfrak{m} \cap \mathfrak{k}})$. The other condition in Lemma 5.2 that needs checking is $J_H(g, \mu + \nu, a, \delta) = 0$, for every $(g, \nu, a, \delta) \in (G \times \mathfrak{g}^* \times B \times U) \cap J_H^{-1}(0)$; this follows easily from the fact that $\mu|_{\mathfrak{h}} = 0$. It is easily checked that σ has an inverse given by $(g, \mu + \nu, a + c, \alpha + \delta) \mapsto (g, \nu, a, \delta)$, where $a \in B$ and $c \in B^\perp$.

Part of that same argument, namely the fact that $\Gamma_\delta^*(\nu|_{\mathfrak{m} \cap \mathfrak{k}} - a \diamond_{\mathfrak{m} \cap \mathfrak{k}} \delta) = -c$ for any $(g, \mu + \nu, a + c, \alpha + \delta)$ in the range of σ , also proves that σ is the unique function, with the given domain and range, of the form $(g, \nu, a, \delta) \mapsto (g, \mu + \nu, a + c, \alpha + \delta)$ for $c \in B^\perp$.

We now show that σ is a diffeomorphism. Note that its domain is a submanifold of $T^*(G \times B)$, being an open subset of a level set of the momentum map of a free action; similarly the range of σ is a submanifold of $T^*(G \times A)$. Since the image of Γ_δ^* is B^\perp , its derivative is always in B^\perp , so for any (g, ν, a, δ) in the domain of σ , and any tangent vector $(\dot{g}, \dot{\nu}, \dot{a}, \dot{\delta}) \in \mathfrak{g} \times \mathfrak{g}^* \times B \times B^*$, we have $T_{(g,\nu,a,\delta)}\sigma(\dot{g}, \dot{\nu}, \dot{a}, \dot{\delta}) = (\dot{g}, \dot{\nu}, \dot{a} + \dot{c}, \dot{\delta})$ for some $\dot{c} \in B^\perp$. It is clear from this formula that σ is an immersion. But any bijective immersion is a diffeomorphism (see Lemma 5.5 below).

We next show that σ is presymplectic. The canonical symplectic forms on the domain and codomain have the same formula,

$$\Omega(g, \nu, a, \delta) \left(\left(\dot{g}_1, \dot{\nu}_1, \dot{a}_1, \dot{\delta}_1 \right), \left(\dot{g}_2, \dot{\nu}_2, \dot{a}_2, \dot{\delta}_2 \right) \right) = \langle \dot{g}_1, \dot{\nu}_2 \rangle - \langle \dot{g}_2, \dot{\nu}_1 \rangle + \langle \nu, [\dot{g}_1, \dot{g}_2] \rangle + \langle \dot{a}_1, \dot{\delta}_2 \rangle - \langle \dot{a}_2, \dot{\delta}_1 \rangle,$$

In calculating

$$\Omega(\sigma(g, \nu, a, \delta)) \left(\left(\dot{g}_1, \dot{\nu}_1, \dot{a}_1 + \dot{c}_1, \dot{\delta}_1 \right), \left(\dot{g}_2, \dot{\nu}_2, \dot{a}_2 + \dot{c}_2, \dot{\delta}_2 \right) \right),$$

with $\dot{c}_1, \dot{c}_2 \in B^\perp$ and $\dot{\delta}_1, \dot{\delta}_2 \in B^*$ the only part containing the \dot{c}_i s is $\langle \dot{c}_1, \dot{\delta}_2 \rangle - \langle \dot{c}_2, \dot{\delta}_1 \rangle$, which equals zero. This shows that σ is presymplectic.

It is clear that σ is G -equivariant. The H -equivariance of σ follows from the H -equivariance of Γ_δ^* and the H -invariance of μ and α .

Since σ is H -equivariant, the map $\bar{\sigma}$ (defined above) is well-defined. It is clearly G -equivariant. Now, $K \cdot (\text{Im } \sigma) = (G \times \mathfrak{g}^* \times A \times (\alpha + U)) \cap J_K^{-1}(0)$, which is an open subset of $J_K^{-1}(0)$. Since $\text{Im } \bar{\sigma} = \varphi(\text{Im } \sigma) = \varphi(K \cdot \text{Im } \sigma)$, this implies that $\text{Im } \bar{\sigma}$ is open. Hence $\bar{\sigma}$ is a surjective submersion onto an open subset of $T^*(G \times_K A)$. For injectivity, suppose $\bar{\sigma}(\psi(w_1)) = \bar{\sigma}(\psi(w_2))$, which is equivalent to $\varphi(\sigma(w_1)) = \varphi(\sigma(w_2))$. By definition of φ , this implies that $\sigma(w_1) = k \cdot \sigma(w_2)$ for some $k \in K$. If the A^* coordinates of w_1 and w_2 are δ_1 and δ_2 , this implies that $\alpha + \delta_1 = k \cdot (\alpha + \delta_2)$. Recall that U was chosen so that the map $t : K \times_H U \rightarrow A^*$, $[k, \delta]_H \mapsto k \cdot (\alpha + \delta)$, is injective. Thus $[e, \delta_1]_H = [k, \delta_2]_H$, which implies $k \in H$. The H -equivariance of σ implies that $\sigma(w_1) = \sigma(k \cdot w_2)$, which implies $w_1 = k \cdot w_2$, since we have shown that σ is injective. Thus $\psi(w_1) = \psi(k \cdot w_2) = \psi(w_2)$, which proves injectivity of $\bar{\sigma}$. Therefore $\bar{\sigma}$ is a bijective submersion, and hence an embedding, onto an open subset of $T^*(G \times_K A)$. It is symplectic since σ is presymplectic; in fact, this is an application of Lemma 3.4 at each base point. ■

Remark 5.4 *The reason for the notation Γ_δ^* is the following: if H is normal in K then there is a free action of K/H on $K \cdot (\alpha + U) \subset A^*$. The Riemannian metric defines a connection 1-form $T(K \cdot (\alpha + U)) \rightarrow (\mathfrak{k}/\mathfrak{h}) \cong (\mathfrak{m} \cap \mathfrak{k})$ on the principal bundle $K \cdot (\alpha + U) \rightarrow K \cdot (\alpha + U) / (K/H)$, defined by orthogonal projection onto the vertical fibre followed by the inverse of the infinitesimal generator map. We re-package this connection 1-form as a map $K \cdot (\alpha + U) \rightarrow L(A^*, \mathfrak{m} \cap \mathfrak{k})$ and compose with the shift map $(k, \delta) \mapsto k \cdot (\alpha + \delta)$, giving the map*

$$\Gamma : K \times U \longrightarrow L(A^*, \mathfrak{m} \cap \mathfrak{k})$$

defined by

$$\Gamma(k, \delta)(k \cdot (\xi \cdot (\alpha + \delta) + \varepsilon)) = \xi$$

for every $\xi \in \mathfrak{m} \cap \mathfrak{k}$ and $\varepsilon \in (\mathfrak{k} \cdot \alpha)^\perp$. Define $\Gamma^* : K \times U \rightarrow L((\mathfrak{m} \cap \mathfrak{k})^*, A)$ by $\Gamma^*(k, \delta) = (\Gamma(k, \delta))^*$. Then for every δ , the map $\Gamma^*(e, \delta)$ equals Γ_δ^* as defined in the above lemma. The proof that $\Gamma^*(e, \delta) \in L((\mathfrak{m} \cap \mathfrak{k})^*, B^\perp)$ and not just $L((\mathfrak{m} \cap \mathfrak{k})^*, A)$ is identical to the proof, in the above lemma, that Γ_δ^* is well-defined.

Remark 5.5 *The fact that every bijective immersion is a diffeomorphism (used in the proof of the above lemma) is well known (see [AMR88]); however the following short proof for finite-dimensional manifolds seems not to be. Let $f : M \rightarrow N$ be a bijective immersion, and let m and n be the dimensions of M and N respectively. Since f is an immersion, we have $m \leq n$. If m were strictly less than n then every point in M would be a critical point, which would imply (since f is surjective) that every point in N was a critical value, contradicting Sard's theorem. Hence $m = n$, so f is a local diffeomorphism at every point. Since f is bijection, it is a diffeomorphism.*

The composition $\tau = \bar{\sigma} \circ \bar{\psi} \circ L$ of the map $\bar{\sigma}$ from Lemma 5.3 with $\bar{\psi} \circ L$ from Equation 21 is the G -equivariant embedding

$$\begin{aligned} \tau : G \times_H (\mathfrak{m}^* \times B \times U) &\longrightarrow T^*(G \times_K A) \\ [g, \nu, a, \delta]_H &\longmapsto \varphi(g, \mu + \nu + a \diamond_{\mathfrak{h}} \delta, a - \Gamma_{\delta}^*(\nu|_{\mathfrak{m} \cap \mathfrak{k}} - a \diamond_{\mathfrak{m} \cap \mathfrak{k}} \delta), \alpha + \delta), \end{aligned} \quad (24)$$

where U and Γ_{δ}^* are as in Lemma 5.3. Since τ maps $[e, 0, 0, 0]_H$ to $\varphi(e, \mu, 0, \alpha)$, and its image is an open subset of $T^*(G \times_K A)$, it is a symplectic tube.

Recall that there is a G -equivariant symplectomorphism $T^*s^{-1} : T^*(G \times_K V) \rightarrow T^*Q$, for some neighbourhood V of 0 in A . The composition of τ with T^*s^{-1} will give our final result. Unfortunately, the preimage $\tau^{-1}(T^*(G \times_K V))$ doesn't have a simple description in general, so we can only say that $T^*s^{-1} \circ \tau$ is defined on some neighbourhood of $[e, 0, 0, 0]_H$. However, in the special case $H = K$, the Γ_{δ}^* term disappears, so the domain of $T^*s^{-1} \circ \tau$ is $G \times_H (\mathfrak{m}^* \times (B \cap V) \times U)$. A second special case occurs if the domain of s is the entire space $(G \times_K A)$, which occurs, for example, if $K = G$ and G acts linearly on Q . In this case the domain of $T^*s^{-1} \circ \tau$ is simply $G \times_H (\mathfrak{m}^* \times B \times U)$. We have proven the following:

Theorem 5.6 (Cotangent Bundle Slice Theorem) *Let G be a Lie group acting properly on a manifold Q and by cotangent lifts on T^*Q , which we give the canonical cotangent symplectic form. Let J be the momentum map for the G action, and let $z \in T_q^*Q$ and $\mu = J(z)$. Assume that $G_{\mu} = G$. Let $H = G_q$ and $K = G_z$, and let \mathfrak{h} and \mathfrak{k} be their Lie algebras. Choose an H -invariant metric on \mathfrak{g} and let \mathfrak{m} be the orthogonal complement to \mathfrak{h} . Choose a K -invariant metric on Q , and let $A = (\mathfrak{g} \cdot q)^{\perp}$. By Palais' slice theorem, there exists a K -invariant neighbourhood V of 0 in A such that the map $s : G \times_K V \rightarrow Q$, $[g, a]_K \mapsto g \cdot \exp_q a$, is a G -equivariant diffeomorphism onto a neighbourhood of q . Let J_K be the momentum map for the cotangent lift of the twist action of K on $G \times A$, and let $\varphi : (J_K^{-1}(0) \subset T^*(G \times A)) \rightarrow T^*(G \times_K A)$ be the cotangent bundle reduction map, defined in Theorem 2.3. Let $\alpha = z|_A$ and $B = (\mathfrak{k} \cdot \alpha)^{\circ}$. There exists an H -invariant neighbourhood N of $(0, 0, 0)$ in $\mathfrak{m}^* \oplus B \oplus B^*$ such that the map*

$$\begin{aligned} T^*s^{-1} \circ \bar{\sigma} \circ \bar{\psi} \circ L : G \times_H N &\longrightarrow T^*Q \\ [g, \nu, a, \delta]_H &\longmapsto T^*s^{-1} \circ \varphi(g, \mu + \nu + a \diamond_{\mathfrak{h}} \delta, a - \Gamma_{\delta}^*(\nu|_{\mathfrak{m} \cap \mathfrak{k}} - a \diamond_{\mathfrak{m} \cap \mathfrak{k}} \delta), \alpha + \delta), \end{aligned}$$

with Γ_{δ}^* as in Lemma 5.3, is a symplectic tube around z .

If $H = K$ or $V = A$ then N may be taken to equal $\mathfrak{m}^* \times (B \cap V) \times U$, where $U \subset B^*$ is chosen as in Lemma 5.3.

Remark 5.7 *There are three new aspects of this result, when compared with the general Hamiltonian slice theorem. First, the symplectic tube is explicitly constructed, up to the cotangent lift of a Riemannian exponential on the configuration space. Second, we have used the cotangent-bundle-specific splitting $N_s \cong T^*B$ in the model space. Third, the tube has a uniqueness property; see Lemma 5.3.*

Remark 5.8 *This result depends crucially on the condition $G_\mu = G$, for the following reasons. The isomorphism $N_s \cong T^*B$ depends on $G_\mu = G$ (see Corollary 4.10) and the isomorphism $G \times \mathfrak{g}_\mu^* \times N_s \cong T^*(G \times B)$ depends on $N_s \cong T^*B$ and also requires $\mathfrak{g}_\mu^* = \mathfrak{g}^*$. The condition $G_\mu = G$ is used twice in the construction of σ : in the splitting $\mathfrak{k} = \mathfrak{m} \cap \mathfrak{k} \oplus \mathfrak{h}$; and in the application of Palais' slice theorem to the K action on A^* , where it is required that $K_\alpha = H$. Finally, $G_\mu = G$ is needed to guarantee that the map σ is symplectic, since this map involves a shift by μ (see the last paragraph of the proof of Lemma 5.3).*

When computing the symplectic tube in the cotangent bundle slice theorem in an example, it is easiest to compute the composition $T^*s^{-1} \circ \varphi$ directly, using the formula

$$\langle T^*s^{-1} \circ \varphi(g, \nu, a, \beta), T(s \circ \pi_K)(g, \xi, a, \dot{a}) \rangle = \langle \nu, \xi \rangle + \langle \beta, \dot{a} \rangle,$$

which follows directly from the definitions of the cotangent lift and the map φ . Since the kernel of $T(s \circ \pi_K)$ is $\mathfrak{k} \cdot (G \times A)$, all elements of TQ can be written as $T(s \circ \pi_K)(g, \xi_\perp, a, \dot{a})$ for some $\xi_\perp \in \mathfrak{k}^\perp$. Note that, when $\xi_\perp \in \mathfrak{k}^\perp$, the \mathfrak{k}^* component of ν is irrelevant in the above equation, and in particular, the term $a \diamond_{\mathfrak{h}} \delta$ in the formula in the cotangent bundle slice theorem is irrelevant.

A particularly simple case is when G acts linearly in a vector space Q and $K = G$. In this case, $A = T_q Q \cong Q$, and all elements of TQ can be written as $T(s \circ \pi_K)(g, 0, a, \dot{a})$. Recalling that for linear actions, $s \circ \pi_K(g, a) = g \cdot (q + a)$, and identifying A with Q , we have $T(s \circ \pi_K)(g, 0, a, \dot{a}) = (g \cdot (q + a), g \cdot \dot{a})$. So the above equation becomes

$$\langle T^*s^{-1} \circ \varphi(g, \nu, a, \beta), (g \cdot (q + a), g \cdot \dot{a}) \rangle = \langle \beta, \dot{a} \rangle,$$

for all $\dot{a} \in Q$, which is equivalent to

$$T^*s^{-1} \circ \varphi(g, \nu, a, \beta) = (g \cdot (q + a), g \cdot \beta). \quad (25)$$

An alternative construction We now give an alternative formulation and proof of Theorem 5.6. The new construction is more elegant but less concrete. We will produce another G -equivariant local symplectomorphism from $G \times_H (\mathfrak{m}^* \times N_s)$ to T^*Q taking $[e, 0, 0]_H$ to z , and then show that it is the same as the one in Theorem 5.6.

We retain all of the definitions from earlier in this section, as well as the assumption $G_\mu = G$. We have seen that $G \times_H (\mathfrak{m}^* \times N_s)$ is isomorphic to $T^*(G \times_H B)$, so that it suffices to find a G -equivariant local symplectomorphism from $T^*(G \times_H B)$ to $T^*(G \times_K A)$ taking $\psi(e, 0, 0, 0)$ to $\varphi(e, \mu, 0, \alpha)$. It is natural to consider the cotangent lift of some G -equivariant diffeomorphism from $G \times_H B$ to $G \times_K A$, since cotangent lifts are automatically symplectic. However, the cotangent lift of any map from $G \times_H B$ to $G \times_K A$ must map

$\psi(e, 0, 0, 0)$, which is in the zero section of $T^*(G \times_H B)$, to some element of the zero section of $T^*(G \times_K A)$, i.e., an element of the form $\varphi(g, 0, a, 0)$; but the target point $\varphi(e, \mu, 0, \alpha)$ is in general not of this form. We might try a momentum shift, but note that the shift $(g, \nu, a, \delta) \mapsto (g, \mu + \nu, a, \alpha + \delta)$ need not preserve $J_K^{-1}(0)$ (see Lemma 5.2), so the “map” $\varphi(g, \nu, a, \delta) \mapsto \varphi(g, \mu + \nu, a, \alpha + \delta)$ is ill-defined.

The idea of using cotangent lifts can be made to work, by “switching the roles of A and A^* ”: modelling $G \times_H (\mathfrak{m}^* \times N_s)$ as $T^*(G \times_H B^*)$ instead of $T^*(G \times_H B)$, and T^*Q as $T^*(G \times_K A^*)$ instead of $T^*(G \times_K A)$. The advantages of this approach will be: (i) $z \in T^*Q$ will correspond to a point in the zero section of $T^*(G \times_K A^*)$; and (ii) there is a simple local diffeomorphism from $G \times_H B^*$ to $G \times_K A^*$, namely $[g, \delta]_H \rightarrow [g, \alpha + \delta]_K$ (see Lemma 5.10.)

Our starting point is the isomorphism in the following lemma, which is easily verified.

Lemma 5.9 *Let G act linearly on a vector space W and by cotangent lifts on T^*W . With respect to the inverse dual action of G on W^* and the corresponding cotangent lifted action on T^*W^* , the map*

$$\begin{aligned} \chi : T^*W \cong W \oplus W^* &\longrightarrow W^* \oplus W \cong T^*W^* \\ (a, \alpha) &\longmapsto (\alpha, -a) \end{aligned}$$

is a G -equivariant symplectomorphism, with respect to the standard symplectic forms. If J and J_ are the standard momentum maps for the G actions on T^*W and T^*W^* respectively, then $J_* \circ \chi = J$, and in particular, $J_*^{-1}(0) = \chi(J^{-1}(0))$.*

It follows that

$$\begin{aligned} \chi_0 : T^*(G \times A) \cong G \times \mathfrak{g}^* \times A \times A^* &\longrightarrow G \times \mathfrak{g}^* \times A^* \times A \cong T^*(G \times A^*) \\ (g, \nu, a, \alpha) &\longmapsto (g, \nu, \alpha, -a) \end{aligned}$$

is symplectic with respect to the canonical symplectic forms, and that $\chi_0(J_K^{-1}(0)) = J_{K,*}^{-1}(0)$, where $J_{K,*}$ is the momentum map of the cotangent lift of the twist action of K on $G \times A^*$. Also, χ_0 is clearly G -equivariant. Applying point cotangent bundle reduction to both sides, χ_0 induces a G -equivariant symplectomorphism

$$\bar{\chi}_0 : T^*(G \times_K A) \longrightarrow T^*(G \times_K A^*).$$

By similar reasoning, the symplectic isomorphism

$$\begin{aligned} \chi_Z : T^*(G \times B) \cong G \times \mathfrak{g}^* \times B \times B^* &\longrightarrow G \times \mathfrak{g}^* \times B^* \times B \cong T^*(G \times B^*) \\ (g, \nu, b, \beta) &\longmapsto (g, \nu, \beta, -b) \end{aligned}$$

maps $J_H^{-1}(0)$ to $J_{H,*}^{-1}(0)$, where $J_{H,*}$ is the momentum map for the cotangent-lift of the twist action of H on $G \times B^*$, and induces a G -equivariant symplectomorphism

$$\bar{\chi}_0 : T^*(G \times_H B) \longrightarrow T^*(G \times_H B^*).$$

Thus, in order to find a G -equivariant local diffeomorphism from $T^*(G \times_H B)$ to $T^*(G \times_K A)$ that maps $\psi(e, 0, 0, 0)$ to $\varphi(e, 0, 0, \alpha)$, it suffices to find one, call it τ_2 , from $T^*(G \times_H B^*)$ to $T^*(G \times_K A^*)$ that maps $\psi_*(e, 0, 0, 0)$ to $\varphi_*(e, 0, \alpha, 0)$, where ψ_* and φ_* are the maps that appear in cotangent bundle reduction (Theorem 2.3), with domain and range as in the following summary diagram,

$$\begin{array}{ccccccc} J_H^{-1}(0) & \xrightarrow{\chi_Z} & J_{H,*}^{-1}(0) & \longrightarrow & J_{K,*}^{-1}(0) & \xrightarrow{\bar{\chi}_0^{-1}} & J_K^{-1}(0) \\ \downarrow \psi & & \downarrow \psi_* & & \downarrow \varphi_* & & \downarrow \varphi \\ T^*(G \times_H B) & \xrightarrow{\bar{\chi}_Z} & T^*(G \times_H B^*) & \xrightarrow{\tau_2} & T^*(G \times_K A^*) & \xrightarrow{\bar{\chi}_0^{-1}} & T^*(G \times_K A). \end{array}$$

The map τ_2 will be the cotangent lift of the diffeomorphism in the following lemma.

Lemma 5.10 *There exists an H -invariant neighbourhood U of α in $B^* = (\mathfrak{k} \cdot \alpha)^\perp$ such that the map*

$$\begin{aligned} F : G \times_H U &\longrightarrow G \times_K (K \cdot (\alpha + U)) \subset G \times_K A^* \\ [g, \delta]_H &\longmapsto [g, \alpha + \delta]_K \end{aligned}$$

is a G -equivariant diffeomorphism of G -invariant neighbourhoods of $[e, 0]_H$ and $[e, \alpha]_K$.

Proof. By Palais' slice theorem for linear actions (Theorem 2.5), there exists an H -invariant neighbourhood U of 0 in $(\mathfrak{k} \cdot \alpha)^\perp$ such that the tube

$$\begin{aligned} K \times_H U &\longrightarrow K \cdot (\alpha + U) \subset A^* \\ [k, \delta]_H &\longmapsto k \cdot (\alpha + \delta) \end{aligned}$$

is a K -invariant diffeomorphism. It follows that the map

$$\begin{aligned} G \times_K (K \times_H U) &\longrightarrow G \times_K (K \cdot (\alpha + U)) \\ [g, [k, \delta]_H]_K &\longmapsto [g, k \cdot (\alpha + \delta)]_K \end{aligned}$$

is a G -equivariant diffeomorphism. It thus suffices to show that the following map is a G -equivariant diffeomorphism,

$$\begin{aligned} G \times_H U &\longrightarrow G \times_K (K \times_H U) \\ [g, \delta]_H &\longmapsto [g, [e, \delta]_H]_K \end{aligned}$$

This is not hard to verify; a proof appears in [OR04]. ■

Note that the definition of U is the same as in Lemma 5.3.

Let F be as in the previous lemma. Its cotangent lift is the G -equivariant symplectomorphism

$$T^*F^{-1} : T^*(G \times_H U) \longrightarrow T^*(G \times_K (K \cdot (\alpha + U))).$$

Since F maps $[e, 0]_H$ to $[e, \alpha]_K$, it follows from the definitions of ψ_* and φ_* that T^*F^{-1} maps $\psi_*(e, 0, 0, 0)$ to $\varphi_*(e, 0, \alpha, 0)$. The composition $\bar{\chi}_0^{-1} \circ T^*F^{-1} \circ \bar{\chi}_Z$ maps $\psi_*(e, 0, 0, 0)$ to $\varphi_*(e, 0, \alpha, 0)$. We compose this with the shift map

$$\begin{aligned} \bar{\Sigma}_{(\mu, 0)} : T^*(G \times_K A) &\longrightarrow T^*(G \times_K A) \\ \psi(g, \nu, a, \delta) &\longmapsto \psi(g, \mu + \nu, a, \delta) \end{aligned}$$

which is easily shown to be a G -equivariant symplectomorphism, by an argument similar to that in Lemma 5.1. The composition $\bar{\Sigma}_{(\mu, 0)} \circ \bar{\chi}_0^{-1} \circ T^*F^{-1} \circ \bar{\chi}_Z$ maps $\psi_*(e, 0, 0, 0)$ to $\varphi_*(e, \mu, 0, \alpha)$. Composing with $\bar{\psi} \circ L$, defined in Equation 21, gives a map

$$\bar{\Sigma}_{(\mu, 0)} \circ \bar{\chi}_0^{-1} \circ T^*F^{-1} \circ \bar{\chi}_Z \circ \bar{\psi} \circ L : G \times_H (\mathfrak{m}^* \oplus B \oplus B^*) \longrightarrow T^*(G \times_K V)$$

taking $[e, 0, 0, 0]_H$ to $\varphi_*(e, 0, \alpha, 0)$. Finally, we compose with $T^*s^{-1} : T^*(G \times_K V) \rightarrow T^*Q$, which forces us to restrict the domain of the composition. The result is an alternative version of the cotangent bundle slice theorem (Theorem 5.6):

Theorem 5.11 *Under the conditions of the Theorem 5.6, there exists an H -invariant neighbourhood N of $(0, 0, 0)$ in $\mathfrak{m}^* \oplus B \oplus B^*$ such that the map*

$$T^*s^{-1} \circ \bar{\Sigma}_{(\mu, 0)} \circ \bar{\chi}_0^{-1} \circ T^*F^{-1} \circ \bar{\chi}_Z \circ \bar{\psi} \circ L : G \times_H N \longrightarrow T^*Q$$

(defined above) is a symplectic tube around z .

We will now show that the symplectic tubes in Theorems 5.6 and 5.11 are the same. To do this, it suffices to show that $\bar{\Sigma}_{(\mu, 0)} \circ \bar{\chi}_0^{-1} \circ T^*F^{-1} \circ \bar{\chi}_Z = \bar{\sigma}$, or equivalently,

$$\bar{\chi}_0 \circ \bar{\Sigma}_{(\mu, 0)}^{-1} \circ \bar{\sigma} \circ \bar{\chi}_Z^{-1} = T^*F^{-1}. \quad (26)$$

It is straight-forward to check that

$$\bar{\chi}_0 \circ \bar{\Sigma}_{(\mu, 0)}^{-1} \circ \bar{\sigma} \circ \bar{\chi}_Z^{-1}(\psi_*(g, \nu, \delta, a)) = \varphi_*(g, \nu, \alpha + \delta, a + \Gamma_\delta^*(\rho|_{\mathfrak{m} \cap \mathfrak{t}} + a \diamond_{\mathfrak{m} \cap \mathfrak{t}} \delta)), \quad (27)$$

for every $\psi_*(g, \nu, \delta, a) \in T^*(G \times_H U)$.

To compute T^*F^{-1} , let U and F be as above, and define $f : G \times U \rightarrow G \times (\alpha + U)$ by $f(g, \delta) = (g, \alpha + \delta)$. It is clear that the following diagram commutes,

$$\begin{array}{ccc} G \times U & \xrightarrow{f} & G \times (\alpha + U) \\ \pi_H \downarrow & & \downarrow \pi_K \\ G \times_H U & \xrightarrow{F} & G \times_K (K \cdot (\alpha + U)) \end{array}$$

where π_H and π_K are restrictions of the canonical projections. Since F is invertible, we have $\pi_H = F^{-1} \circ \pi_K \circ f$. The surjectivity of F also implies that every element of $T(G \times_K (K \cdot (\alpha + U)))$ can be expressed as $T(\pi_K \circ f)(g, \xi, \delta, \epsilon) = T\pi_K(g, \xi, \alpha + \delta, \epsilon)$ for

some $(g, \xi, \delta, \epsilon) \in T(G \times U)$. Hence we can compute T^*F^{-1} as follows: for any $(g, \nu, \delta, a) \in T^*(G \times U) \cap J_{K,*}^{-1}(0)$ and any $(g, \xi, \delta, \epsilon) \in T(G \times U)$,

$$\begin{aligned} & \langle T^*F^{-1} \circ \psi_*(g, \nu, \delta, a), T\pi_K(g, \xi, \alpha + \delta, \epsilon) \rangle \\ &= \langle \psi_*(g, \nu, \delta, a), T(F^{-1} \circ \pi_K \circ f)(g, \xi, \delta, \epsilon) \rangle \\ &= \langle \psi_*(g, \nu, \delta, a), T\pi_H(g, \xi, \delta, \epsilon) \rangle \\ &= \langle \nu, \xi \rangle + \langle a, \epsilon \rangle. \end{aligned} \quad (28)$$

Now we make the corresponding computation with the right-hand side of Equation 27, namely $\varphi_*(g, \nu, \alpha + \delta, a + \Gamma_\delta^*(\rho|_{\mathfrak{m} \cap \mathfrak{k}} + a \diamond_{\mathfrak{m} \cap \mathfrak{k}} \delta))$. Since $\Gamma_\delta^*(\rho|_{\mathfrak{m} \cap \mathfrak{k}} + a \diamond_{\mathfrak{m} \cap \mathfrak{k}} \delta) \in B^\perp$, which annihilates $\epsilon \in U \subset B^*$, we have,

$$\begin{aligned} & \langle \varphi_*(g, \nu, \alpha + \delta, a + \Gamma_\delta^*(\rho|_{\mathfrak{m} \cap \mathfrak{k}} + a \diamond_{\mathfrak{m} \cap \mathfrak{k}} \delta)), T\pi_K(g, \xi, \alpha + \delta, \epsilon) \rangle \\ &= \langle \nu, \xi \rangle + \langle a + \Gamma_\delta^*(\rho|_{\mathfrak{m} \cap \mathfrak{k}} + a \diamond_{\mathfrak{m} \cap \mathfrak{k}} \delta), \epsilon \rangle \\ &= \langle \nu, \xi \rangle + \langle a, \epsilon \rangle. \end{aligned} \quad (29)$$

The calculations in Equations 28 and 29 prove that

$$T^*F^{-1}(\psi_*(g, \nu, \delta, a)) = \varphi_*(g, \nu, \alpha + \delta, a + \Gamma_\delta^*(\rho|_{\mathfrak{m} \cap \mathfrak{k}} + a \diamond_{\mathfrak{m} \cap \mathfrak{k}} \delta)), \quad (30)$$

which, together with Equation 27, proves Equation 26. Thus we have shown:

Theorem 5.12 *The symplectic tubes in Theorems 5.6 and 5.11 are identical.*

Remark 5.13 *Using Lemma 5.2, it can be easily checked that $\Gamma_\delta^*(\rho|_{\mathfrak{m} \cap \mathfrak{k}} + a \diamond_{\mathfrak{m} \cap \mathfrak{k}} \delta)$ is the unique element $c \in B^\perp$ such that $(g, \nu, \alpha + \delta, a + c) \in J_{K,*}^{-1}(0)$. Thus the calculations in Equations 28 and 29 show that the formula in Equation 30 is the unique one expressing $T^*F^{-1}(\psi_*(g, \nu, \delta, a))$ as the φ_* -image of an element of $T_{(g, \alpha + \delta)}^*(G \times A)$.*

Example ($SO(3)$ acting on $T^*\mathbf{R}^3$) We conclude this section with a calculation in a simple example of the symplectic tube in the cotangent bundle slice theorem (Theorem 5.6). Consider $G = SO(3)$ acting on in the standard way on $Q = \mathbf{R}^3$, and by cotangent lifts on $T^*\mathbf{R}^3$. The momentum map is $\mu = J(q, p) = q \times p$. This one example is actually many, because we can vary the point $z = (q, p) \in T^*\mathbf{R}^3$ around which we construct a tube. In order to apply Theorem 5.6 we require $G_\mu = G$; in this case, the coadjoint action of $SO(3)$ is such that this condition is satisfied only at $\mu = 0$. Thus q and p must be parallel, or at least one of them must be zero. We will present the case $q = 0$ and $p \neq 0$, and then state without details the results of similar calculations for the other cases. We will implicitly use the Euclidean inner product in several places, to define orthogonal complements and to identify spaces with their duals.

Assume $q = 0$ and $p \neq 0$. Without loss of generality $z = (q, p) = ((0, 0, 0), (\lambda, 0, 0))$ for some $\lambda \neq 0$. We have $K = G = SO(3)$, and H is the circle group of rotations around the x -axis. Since G fixes q , we have $A = (\mathfrak{g} \cdot q)^\perp = \mathbf{R}^3$. Also, $\alpha = z|_A = (\lambda, 0, 0) \in A^*$.

Since $K = SO(3)$, the group orbit $K \cdot \alpha$ is the sphere of radius λ , so $B := (\mathfrak{k} \cdot \alpha)^\circ$ is the x -axis (identifying $(\mathbf{R}^3)^*$ with \mathbf{R}^3); the space B^* is also the x -axis. We make the standard identification

$$\begin{aligned} \mathfrak{g} = \mathfrak{so}(3) &\longrightarrow \mathbf{R}^3 \\ \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} &\longmapsto (\xi_1, \xi_2, \xi_3). \end{aligned}$$

Since H is the group of rotations around the x -axis, its Lie algebra, \mathfrak{h} , is the x -axis. Now $\mathfrak{m} = \mathfrak{h}^\perp$ so, using the Euclidean metric again, we can identify \mathfrak{m} and \mathfrak{m}^* with the yz -plane. We now calculate the map τ from Equation 24, beginning with the subset $U \subset B^*$ and the map Γ_δ^* in Lemma 5.3. Let $U = (-\lambda, \infty) \times \{(0, 0)\} \subset B^*$; this is the largest neighbourhood in B^* such that the tube $t : K \times_H U \longrightarrow A^*, [g, \delta]_H \longmapsto g \cdot (\alpha + \delta)$ is injective. Since $K = G$, we have $\mathfrak{m} \cap \mathfrak{k} = \mathfrak{m}$. The map $\Gamma_\delta^* : (\mathfrak{m} \cap \mathfrak{k})^* \rightarrow B^\perp$ is defined by

$$\langle \Gamma_\delta^*(\nu), \xi \cdot (\alpha + \delta) + \varepsilon \rangle = \langle \nu, \xi \rangle,$$

for every $\xi \in \mathfrak{m} \cap \mathfrak{k}, \nu \in (\mathfrak{m} \cap \mathfrak{k})^*, \delta \in U$ and $\varepsilon \in B^*$. Let $\nu = (0, \nu_2, \nu_3), \xi = (0, \xi_2, \xi_3), \delta = (\delta_1, 0, 0)$ and $\varepsilon = (\varepsilon_1, 0, 0)$, and recall that $\alpha = (\lambda, 0, 0)$. Then

$$\xi \cdot (\alpha + \delta) + \varepsilon = \xi \cdot (\lambda + \delta_1, 0, 0) + (\varepsilon_1, 0, 0) = (\varepsilon_1, \xi_3(\lambda + \delta_1), -\xi_2(\lambda + \delta_1)),$$

so $\langle \Gamma_\delta^*(\nu), (\varepsilon_1, \xi_3(\lambda + \delta_1), -\xi_2(\lambda + \delta_1)) \rangle = \langle \nu, \xi \rangle$. It follows that $\Gamma_\delta^*(\nu) = \left(0, \frac{\nu_3}{\lambda + \delta_1}, -\frac{\nu_2}{\lambda + \delta_1}\right)$. Now $a \diamond_{\mathfrak{g}} \delta = J_K(a, \delta) = a \times \delta$. Since B is the x -axis and $U \subset B^*$, we have $a \times \delta = 0$ for all $a \in B$ and $\delta \in U$. Putting these calculations together,

$$\tau[g, \nu, ((a_1, 0, 0)), (\delta_1, 0, 0)]_H = \varphi\left(g, \nu, \left(a_1, \frac{\nu_3}{\lambda + \delta_1}, -\frac{\nu_2}{\lambda + \delta_1}\right), (\lambda + \delta_1, 0, 0)\right).$$

The symplectic tube in Theorem 5.6 is $T^*s^{-1} \circ \tau$. Since $K = G = SO(2)$, we know from Equation 25 that $T^*s^{-1} \circ \varphi(g, \nu, a, \beta) = (g \cdot a, g \cdot \beta)$. So we obtain the following symplectic tube for the G action around z (dropping the subscript-1's):

$$\begin{aligned} T^*s^{-1} \circ \tau : G \times_H (\mathfrak{m}^* \times B \times U) &\longrightarrow T^*\mathbf{R}^3 \\ [g, (\nu_1, \nu_2, \nu_3), (a, 0, 0), (\delta, 0, 0)]_H &\longmapsto \left(g \cdot \left(a, \frac{\nu_3}{\lambda + \delta}, -\frac{\nu_2}{\lambda + \delta}\right), g \cdot (\lambda + \delta, 0, 0)\right) \end{aligned}$$

The other nontrivial subcase of the $Q = \mathbf{R}^3, G = SO(3)$ example occurs when $q \neq 0$ and $p \parallel q$; it turns out that it makes no difference whether or not $p = 0$. In this case, $H = K \cong SO(2)$. Since $H = K$, the map Γ_δ^* is trivial. However, since K is neither G nor $\{e\}$, the calculation of $T^*s^{-1} \circ \varphi$ is nontrivial, though not difficult. For brevity, we state only the final result for this case: if $z = ((\kappa, 0, 0), (\lambda, 0, 0))$, then \mathfrak{m}^* may be identified with the yz -plane, and A and B with the x -axis. The map

$$\begin{aligned} T^*s^{-1} \circ \tau : G \times_H (\mathfrak{m}^* \times (-\kappa, \infty) \times B^*) &\longrightarrow T^*\mathbf{R}^3 \\ [g, (\nu_1, \nu_2, \nu_3), (a, 0, 0), (\delta, 0, 0)]_H &\longmapsto \left(g \cdot (\kappa + a, 0, 0), g \cdot \left(\lambda + \delta, \frac{\nu_3}{\kappa + a}, -\frac{\nu_2}{\kappa + a}\right)\right) \end{aligned}$$

is a symplectic tube for the G action around z .

The only remaining subcase is $q = p = 0$. In this case, $H = K = SO(3)$, $A = B = \mathbf{R}^3$ and $U = B^*$. The map $T^*s^{-1} \circ \tau : G \times_G (A \times A^*) \rightarrow T^*A$ is the trivial one $[g, a, \delta]_G \mapsto (a, \delta)$.

6 Conclusion

We have investigated the local structure of a cotangent bundle with a Lie group of cotangent-lifted symmetries. We proved a “tangent-level” commuting reduction result, Theorem 3.5, and then used it in Section 4 to analyse the symplectic normal space. In two special cases, we arrived at splittings of the symplectic normal space. One of these splittings, Corollary 4.10, applies whenever the configuration isotropy group K is contained in the momentum isotropy group G_μ . We noted that this occurs whenever K is normal in G , for example when G is abelian. The splitting in Corollary 4.10 generalises the one given for free actions by Montgomery et al. [MMR84]. The conditions on the other splitting, in Corollary 4.13, are satisfied by all relative equilibria of simple mechanical systems. In both of these special cases, the new splitting leads to a refinement of the reconstruction equations (bundle equations), as explained at the end of Section 4. We also noted in Section 4 two cotangent-bundle-specific local normal forms for the symplectic reduced space, in Theorem 4.8 and Remark 4.15.

Our main result is a cotangent bundle slice theorem, Theorem 5.6, which applies at all points with fully isotropic momentum values, $G_\mu = G$. This theorem extends the Hamiltonian slice theorem of Marle, Guillemin and Sternberg in three ways. First, it is constructive, apart from the use of the cotangent lift of a Riemannian exponential. Second, it includes a cotangent-bundle-specific splitting of the symplectic normal space (a special case of one of the first of the splittings described in the previous paragraph). Third, our construction has a uniqueness property, contained in Lemma 5.3. In Theorems 5.11 and 5.12, we gave an alternative construction of the symplectic tube in the main theorem, showing that it is essentially a cotangent lift of a simple map between certain twisted products. The example presented at the end of the Section 5 shows that our construction is feasible; we believe that this is the first time that symplectic tubes have been computed in an example.

A number of open questions remain, the most salient of which is: what happens when μ is not fully isotropic? We have so far only been able to formulate our cotangent bundle slice theorem for the case of a fully-isotropic momentum value, for reasons summarised in Remark 5.8. Our characterisation of the symplectic normal space N_s is also incomplete in the general case. We have found a splitting of N_s that applies to all relative equilibria of simple mechanical systems, but what about relative equilibria of other systems? Even for the simple mechanical case, what form do the reconstruction equations take if μ is not split?

One possible application of this work is to the problem of singular cotangent bundle reduction (this was in fact our initial motivation for this research). Local normal forms given in Section 4 are a start, but do not address the global structure.

Dynamical applications seem the most promising. To start with, the constructive nature of the cotangent bundle slice theorem should allow us to apply theoretical results on

stability, bifurcations and persistence, such as those referred to in the Introduction, to specific examples. Also, the refinement of the reconstruction equations in the cotangent bundle case may lead to extensions of the theory. In particular, the relationship between our splitting of the symplectic normal space at a relative equilibrium of a simple mechanical system, and the splitting used in the Lagrangian Block Diagonalisation [Lew92] method for testing stability, deserves investigation.

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